TRANSVERSE LINEAR SUBSPACES TO HYPERSURFACES OVER FINITE FIELDS

SHAMIL ASGARLI, LIAN DUAN, AND KUAN-WEN LAI

Abstract. We prove that if $X$ is a smooth hypersurface of degree $d$ in $\mathbb{P}^n$ defined over a finite field $\mathbb{F}_q$ with $q \geq (n - r)d(d - 1)^r$, then there exists an $r$-plane $H \subset \mathbb{P}^n$ defined over $\mathbb{F}_q$ such that $X \cap H$ is smooth. This generalizes a result by Ballico in the case of hypersurfaces. We also study the existence of transverse lines to reduced hypersurfaces.

1. Introduction

The theorem of Bertini asserts that a smooth projective variety $X$ over an infinite field $k$ admits a smooth hyperplane section. By repeatedly applying the theorem, one can obtain a linear section on $X$ of any dimension without extending the ground field $k$.

If $k = \mathbb{F}_q$ is a finite field, then Bertini’s theorem is no longer true in its original form as there are only finitely many hyperplanes in $\mathbb{P}^n$ defined over $\mathbb{F}_q$, and they could all happen to be tangent to $X$. There are at least two remedies in this situation:

1. Instead of intersecting $X$ with hyperplanes, one could allow intersection with hypersurfaces of arbitrary degree. This approach was taken by Poonen in [Poo04], where he proved the existence of a hypersurface $Y$ over $\mathbb{F}_q$ such that $X \cap Y$ is smooth.

2. When $q$ is sufficiently large with respect to $d := \deg(X)$, we still expect Bertini’s theorem to be valid over the field $\mathbb{F}_q$. In this direction, Ballico [Bal03] proved that if $q \geq d(d - 1)^{\dim X}$, then there exists a hyperplane $H$ over $\mathbb{F}_q$ such that $X \cap H$ is smooth.

Applying Ballico’s result inductively, one can obtain a smooth linear section on $X$ over $\mathbb{F}_q$ of any dimension once the inequality $q \geq d(d - 1)^{n-1}$ holds.

In this paper, we study Bertini’s theorem of linear sections on hypersurfaces. Given a hypersurface $X \subset \mathbb{P}^n$, we say a linear subspace $H \subset \mathbb{P}^n$ is transverse to $X$ if it is disjoint from the singular locus of $X$.

2020 Mathematics Subject Classification 14G15, 14J70, 14N05.
Keywords: hypersurfaces, finite fields, Bertini’s theorem
and the intersection $X \cap H$ is smooth. We will assume $n \geq 3$ throughout the paper. Our first result is to relax Ballico’s bound $d(d - 1)^{n-1}$ to the quadratic bound $q \geq \frac{3}{2}d(d - 1)$ for transverse lines to a reduced hypersurface:

**Theorem 1.1.** Let $X \subset \mathbb{P}^n$ be a reduced hypersurface of degree $d \geq 2$ defined over $\mathbb{F}_q$. Suppose that

$$q \geq \frac{3}{2}d(d - 1).$$

Then there exists an $\mathbb{F}_q$-line $L \subset \mathbb{P}^n$ which is transverse to $X$.

Note that the case $d = 1$ is immediate, as every hyperplane admits a transverse line over $\mathbb{F}_q$. We remark that the case of plane curves was previously investigated in [Asg19], and the case of surfaces in $\mathbb{P}^3$ was the subject of our previous paper [ADL20]. Theorem 1.1 is a consequence of the following:

**Theorem 1.2.** Let $X \subset \mathbb{P}^n$ be a reduced hypersurface of degree $d \geq 2$ over $\mathbb{F}_q$. Then there exists a hyperplane $H \subset \mathbb{P}^n$ over $\mathbb{F}_q$ such that $X \cap H$ is proper and reduced provided that

- $q \geq d(d - 1) + 1$ when $n = 3$.
- $q \geq d$ when $n \geq 4$.

By applying this result inductively, one can obtain a reduced linear section on $X$ over $\mathbb{F}_q$ of any positive dimension once the hypothesis holds. The existence of an $\mathbb{F}_q$-line $L$ which meets a plane curve in reduced sections, i.e. $L$ is a transverse line, is treated in Proposition 2.7. Theorems 1.1 and 1.2 will be proved in Section 2.

Our next result concerns the existence of transverse linear subspaces over $\mathbb{F}_q$ of any dimension:

**Theorem 1.3.** Let $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree $d \geq 3$ defined over $\mathbb{F}_q$ and let $0 \leq r \leq n - 1$. Suppose that

$$q \geq (n - r)d(d - 1)^r.$$

Then there exists an $r$-plane $H \subset \mathbb{P}^n$ over $\mathbb{F}_q$ which is transverse to $X$.

When $r = n - 1$, this is a consequence of Ballico’s result [Bal03]. We remark that when $r = 1$, Theorem 1.3 yields the existence of a transverse $\mathbb{F}_q$-line for $q \geq (n - 1)d(d - 1)$, so the bound given in Theorem 1.1 is stronger in this case.

The hypothesis $d \geq 3$ is assumed in Theorem 1.3, since the case $d = 1$ is immediate, and for $d = 2$, Ballico’s bound $q \geq d(d - 1)^{n-1}$ reduces to $q \geq 2$. By induction on the dimension, it follows that every
smooth quadratic hypersurface over $\mathbb{F}_q$ admits a transverse $r$-plane. Theorem 1.3 will be proved in Section 3.

**Acknowledgments.** The first author was partially supported by NSF grant 1701659, and by a postdoctoral research fellowship from the University of British Columbia.

2. **Transverse lines to reduced hypersurfaces**

In this section, we will prove Theorem 1.1 which we recall below:

**Theorem 2.1.** Let $X \subset \mathbb{P}^n$ be a reduced hypersurface of degree $d \geq 2$ defined over $\mathbb{F}_q$. Suppose that

$$q \geq \frac{3}{2}d(d-1).$$

Then there exists an $\mathbb{F}_q$-line $L \subset \mathbb{P}^n$ which is transverse to $X$.

2.1. **Notation and core machinery.** Consider a hypersurface

$$X = \{ F = 0 \} \subset \mathbb{P}^n$$

defined over $\mathbb{F}_q$. Let us fix a coordinate system $\{x_0, \ldots, x_n\}$ for $\mathbb{P}^n$ and, for the sake of simplicity, let us denote $F_i := \partial F/\partial x_i$ for $0 \leq i \leq n$. Consider the $2 \times (n+1)$ matrix

$$M := \begin{pmatrix} F_0 & \ldots & F_n \\ F_0^q & \ldots & F_n^q \end{pmatrix}$$

For every $0 \leq i < j \leq n$, the maximal minor formed by the $i$-th and the $j$-th columns of this matrix determines a hypersurface

$$D_{ij} := \{ F_i F_j^q - F_i^q F_j = 0 \} \subset \mathbb{P}^n$$

defined over $\mathbb{F}_q$. We will also consider the determinantal variety

$$Z_X := \bigcap_{0 \leq i < j \leq n} D_{ij}.$$

**Remark 2.2.** Consider the polar map induced by $F$:

$$\gamma : \mathbb{P}^n \to (\mathbb{P}^n)^* : [x_0 : \ldots : x_n] \mapsto [F_0 : \ldots : F_n].$$

Note that $\gamma|_X$ is the Gauss map of $X$. Let $\{y_0, \ldots, y_n\}$ be a coordinate system for the dual space $(\mathbb{P}^n)^*$. Observe that the matrix $M$ is the pullback by $\gamma$ of the matrix

$$\begin{pmatrix} y_0 & \ldots & y_n \\ y_0^q & \ldots & y_n^q \end{pmatrix},$$
and the hypersurfaces $D_{ij}$ are the pullbacks of its maximal minors
\[ \{ y_i y_j^q - y_j^q y_i = 0 \} \subset (\mathbb{P}^n)^*. \]
From this one can see that $Z_X \subset \mathbb{P}^n$ is the pullback of the union of $\mathbb{F}_q$-points in $(\mathbb{P}^n)^*$ via $\gamma$.

**Lemma 2.3.** Let $X \subset \mathbb{P}^n$ be a hypersurface and let $P \in X$ be a geometric point. Then $P \in Z_X$ if and only if $P$ is a singular point of $X$ or if the tangent hyperplane $T_P X$ is defined over $\mathbb{F}_q$.

**Proof.** Observe that $P \in Z_X$ if and only if the matrix $M$ has rank one when evaluated at $P$, or equivalently
- $[F_0(P) : \cdots : F_n(P)] = [0 : \cdots : 0]$, or
- $[F_0(P) : \cdots : F_n(P)] \in \mathbb{P}^n(\mathbb{F}_q)$.
These conditions hold if and only if $P$ is a singular point of $X$ or if $T_P X$ is defined over $\mathbb{F}_q$, respectively. \qed

**Lemma 2.4.** Let $X \subset \mathbb{P}^n$ be a reduced hypersurface and consider the determinantal variety $Z_X$ defined with respect to $X$. Then, for every component $X' \subset X$ with $\deg(X') \geq 2$ that is geometrically irreducible, we have $X' \not\subset Z_X$. In particular, there exists $D_{ij}$ such that $X' \cap D_{ij}$ has dimension $n - 2$.

**Proof.** Suppose, to the contrary, that $X' \subset Z_X$. Then each $P \in X'$ is either a singular point of $X$ or has the tangent hyperplane $T_P X$ defined over $\mathbb{F}_q$ by Lemma 2.3. So we obtain the inclusion relation
\[ X' \subset \text{Sing}(X) \cup \bigcup_{H \in (\mathbb{P}^n)^*(\mathbb{F}_q)} (X' \cap H). \]
This is a contradiction as both $\text{Sing}(X)$ and $X' \cap H$ have dimensions strictly less than $\dim(X') = n - 1$, where the latter follows from the hypothesis that $\deg(X') \geq 2$. Therefore, we have $X' \not\subset Z_X$.

For the last statement, notice that $X' \not\subset Z_X$ implies that $X' \not\subset D_{ij}$ for some $0 \leq i < j \leq n$. It follows that $\dim(X' \cap D_{ij}) = n - 2$ since $X'$ is geometrically irreducible. \qed

### 2.2. Transverse lines to reduced plane curves.
Here we prove Theorem 2.1 in the case of plane curves. Later on, we will develop a process to reduce the general case to this special case.

**Lemma 2.5.** Let $C \subset \mathbb{P}^2$ be a reduced and geometrically irreducible curve of degree $d \geq 2$ defined over $\mathbb{F}_q$. Then the number of $\mathbb{F}_q$-lines not transverse to $C$ is bounded by
\[ \frac{1}{2}(d - 1)(3d - 2)(q + 1). \]
Proof. An $\mathbb{F}_q$-line $L$ is not transverse to $C$ either if it passes through a singular point of $C$ or if $L = T_P C$ for some $P \in C$. Since $C$ is geometrically irreducible, the number of singular points of $C$ is at most\[\frac{1}{2}(d-1)(d-2)\]
which can be derived from, for example, [Liu02, §7.5, Proposition 5.4]. Because each singular point has at most $q + 1$ distinct $\mathbb{F}_q$-lines passing through it, this accounts for\[\frac{1}{2}(d-1)(d-2)(q + 1)\]
non-transverse $\mathbb{F}_q$-lines.

To estimate the number of the second type of non-transverse lines, we first note that the condition $L = T_P C$ implies that $T_P C$ is over $\mathbb{F}_q$ and thus $P \in Z_C$ by Lemma 2.3. As $C$ is geometrically irreducible, it intersects some $D_{ij}$ in 0-dimensional scheme by Lemma 2.4. Thus, the number of $\mathbb{F}_q$-lines that arise as $T_P C$ is at most\[C \cdot D_{ij} = \deg(C) \deg(D_{12}) = d(d-1)(q + 1)\]
Consequently, the number of non-transverse $\mathbb{F}_q$-lines to $C$ is at most\[\frac{1}{2}(d-1)(d-2)(q + 1) + d(d-1)(q + 1) = \frac{1}{2}(d-1)(3d-2)(q + 1)\]
as claimed. \qed

Lemma 2.6. Let $C \subset \mathbb{P}^2$ be a reduced curve of degree $d \geq 2$ over $\mathbb{F}_q$. Then the number of $\mathbb{F}_q$-lines not transverse to $C$ is bounded by\[\frac{3}{2} d(d-1)(q + 1) .\]

Proof. Let us decompose $C$ into geometrically irreducible components\[C = C_1 \cup \cdots \cup C_\ell\]
and let $d_i := \deg(C_i)$. For each $\mathbb{F}_q$-line $L$ not transverse to $C$, we have that

(i) $L$ meets $C_i$ non-transversely for some $i$ where $\deg(C_i) \geq 2$, or
(ii) $L$ passes through an intersection point of $C_i$ and $C_j$ for some $i \neq j$. Note that this includes $L$ which meets $C_i$ non-transversely where $\deg(C_i) = 1$. 

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By applying Lemma 2.5 to each component $C_i$ and summing up all the upper bounds, we conclude that the number of lines in (i) is at most

$$\frac{1}{2} \sum_{i=1}^{\ell} (d_i - 1)(3d_i - 2)(q + 1).$$

On the other hand, the number of points in $C_i \cap C_j$ for $i \neq j$ is at most $d_id_j$ by Bezout’s theorem. Since there are at most $(q + 1)$ lines defined over $\mathbb{F}_q$ that passes through a point in $C_i \cap C_j$, the number of lines in (ii) is at most

$$\sum_{i<j} d_id_j(q + 1).$$

By adding up all the contributions above, we obtain that the number of $\mathbb{F}_q$-lines not transverse to $C$ is at most

$$\frac{1}{2} \sum_{i=1}^{\ell} (d_i - 1)(3d_i - 2)(q + 1) + \sum_{i<j} d_id_j(q + 1)$$

Notice that each summand has $q + 1$ as a factor while the rest can be simplified as

$$\frac{1}{2} \sum_{i=1}^{\ell} (d_i - 1)(3d_i - 2) + \sum_{i<j} d_id_j$$

$$= \sum_{i=1}^{\ell} \left( d_i^2 - \frac{5}{2}d_i + 1 \right) + \sum_{i<j} d_id_j$$

$$= \sum_{i=1}^{\ell} \left( d_i^2 - \frac{5}{2}d_i + 1 \right) + \frac{1}{2} \left( \sum_{i=1}^{\ell} d_i^2 + 2 \sum_{i<j} d_id_j \right)$$

$$= \sum_{i=1}^{\ell} d_i^2 - \frac{5}{2}d + \ell + \frac{1}{2}d^2$$

where the last equality uses $d = \sum_{i=1}^{\ell} d_i$. Using the facts that

$$\sum_{i=1}^{\ell} d_i^2 \leq \left( \sum_{i=1}^{\ell} d_i \right)^2 = d^2 \quad \text{and} \quad \ell \leq d,$$
we conclude that the number of $\mathbb{F}_q$-lines not transverse to $C$ is at most
\[
\left( d^2 - \frac{5}{2}d + d + \frac{1}{2}d^2 \right) (q + 1) = \left( \frac{3}{2}d^2 - \frac{3}{2}d \right) (q + 1) = \frac{3}{2}d(d - 1)(q + 1)
\]
as desired. \qed

**Proposition 2.7.** Let $C \subset \mathbb{P}^2$ be a reduced curve of degree $d \geq 2$ defined over $\mathbb{F}_q$. Suppose that
\[
q \geq \frac{3}{2}d(d - 1).
\]
Then there exists an $\mathbb{F}_q$-line in $\mathbb{P}^2$ that is transverse to $C$.

**Proof.** The number of $\mathbb{F}_q$-lines in $\mathbb{P}^2$ is $q^2 + q + 1$, so, by Lemma 2.6, there exists a transverse $\mathbb{F}_q$-line if
\[
q^2 + q + 1 > \frac{3}{2}d(d - 1)(q + 1).
\]
This inequality holds under the hypothesis $q \geq \frac{3}{2}d(d - 1)$. Indeed, we have
\[
q^2 + q + 1 > q^2 + q = q(q + 1) \geq \frac{3}{2}d(d - 1)(q + 1)
\]
which completes the proof. \qed

**Remark 2.8.** There is a different method [AG20, Proposition 2.2] to prove the previous proposition at the cost of slightly stronger hypothesis $q \geq 2d(d - 1)$.

2.3. **Existence of reduced hyperplane sections.** We conclude the proof for Theorem 2.1 in this section. Let $X \subset \mathbb{P}^n$ be a reduced hypersurface of degree $d$ over $\mathbb{F}_q$. The key step in our proof is to find a hyperplane $H \subset \mathbb{P}^n$ over $\mathbb{F}_q$ such that the intersection $X \cap H$ is proper and reduced. This will allow us to reduce the proof inductively to the case of plane curves.

First of all, let us decompose $X$ into geometrically irreducible components
\[
X = X_1 \cup \cdots \cup X_\ell
\]
and let $d_i$ be the degree of $X_i$. After rearranging the indices, we may assume there exists $1 \leq t \leq \ell$ such that

- $d_i = 1$ for $1 \leq i \leq t$,
- $d_i > 1$ for $t + 1 \leq i \leq \ell$.

To attain our goal, we need to estimate the number of hyperplanes $H$ over $\mathbb{F}_q$ which does not satisfy our requirements, namely:
(I) \( X \cap H \) is not proper, that is, \( \dim(X \cap H) = n - 1 \). This implies that \( H = X_i \) for some \( 1 \leq i \leq t \).

(II) \( X \cap H \) is proper but not reduced. This implies that \( X \cap H \) contains a non-reduced and geometrically irreducible component of dimension \( n - 2 \).

For each hyperplane \( H \subset \mathbb{P}^n \) over \( \mathbb{F}_q \), let us define

\[ \mathcal{A}_H := \{ Y \subset X \mid Y \text{ is a non-reduced and geometrically irreducible component of } X \cap H \text{ of dimension } n - 2 \} \]

and then take their union

\[ \mathcal{B} := \bigcup_{H \in (\mathbb{P}^n)^*} \mathcal{A}_H. \]

By definition, for each \( Y \in \mathcal{B} \), a geometric point \( P \in Y \) is either a singular point of \( X \) or satisfies \( T_P X = H \) for some hyperplane \( H \) over \( \mathbb{F}_q \), hence \( P \in Z_X \). Therefore, we have

\[ Y \subset Z_X = \bigcap_{0 \leq i < j \leq n} D_{ij}. \]

Let us decompose \( \mathcal{B} \) into the disjoint union \( \mathcal{B} = \mathcal{B}_1 \sqcup \mathcal{B}_2 \) where

\[ \begin{align*}
\mathcal{B}_1 &= \{ Y \in \mathcal{B} \mid Y \subset X_k \text{ for some } k \text{ with } d_k > 1 \}, \\
\mathcal{B}_2 &= \{ Y \in \mathcal{B} \mid Y \not\subset X_k \text{ for all } k \text{ with } d_k > 1 \}.
\end{align*} \]

Our estimate for the number of \( \mathbb{F}_q \)-hyperplanes of types (I) and (II) is accomplished by estimating the cardinalities of \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \).

**Lemma 2.9.** The cardinality of \( \mathcal{B}_1 \) is bounded by

\[ \sum_{k=t+1}^{\ell} d_k(d-1)(q+1) \]

**Proof.** Let \( Y \in \mathcal{B}_1 \) so that \( Y \subset X_k \) for some \( k \) with \( d_k > 1 \). By Lemma 2.4, there exists \( D_{ij} \) such that \( X_k \cap D_{ij} \) has dimension \( n - 2 \). Hence \( Y \) appears as a component of \( X_k \cap D_{ij} \) due to (2.1).

Let \( s \) denote the number of geometrically irreducible components of \( X_k \cap D_{ij} \) of dimension \( n - 2 \). By Bezout’s theorem, we get

\[ s \leq \deg(X_k \cap D_{ij}) = \deg(X_k) \deg(D_{ij}) = d_k(d-1)(q+1). \]

We conclude that the cardinality of \( \mathcal{B}_1 \) is at most

\[ \sum_{k=t+1}^{\ell} d_k(d-1)(q+1). \]

Notice that the sum starts from \( k = t+1 \) since \( d_k = 1 \) for \( 1 \leq k \leq t \). \( \square \)
Lemma 2.10. The cardinality of $B_2$ is bounded by $\binom{t}{2}$.

Proof. Let $Y \in B_2$ so that $Y \not\subset X_k$ for all $k$ with $d_k > 1$. This implies that $Y \subset X_k \cap H$ where $X_k$ and $H$ are distinct hyperplanes over $\mathbb{F}_q$. Moreover, as $Y$ is non-reduced, it arises from the intersection of two hyperplane components of $X$.

Since $X$ has $t$ distinct hyperplanes as its components, there are at most $\binom{t}{2}$ mutual intersections between them, which gives an upper bound for the cardinality of $B_2$. □

Proposition 2.11. Let $X \subset \mathbb{P}^n$ be a reduced hypersurface of degree $d$ over $\mathbb{F}_q$. Then the number of hyperplanes $H \subset \mathbb{P}^n$ over $\mathbb{F}_q$ such that $X \cap H$ is not proper or non-reduced is bounded by

$$(d - t)(d - 1)(q + 1)^2 + \frac{1}{2}t(t - 1)(q + 1) + 1.$$

Proof. First assume that $t \geq 2$, that is, $X$ contains at least two hyperplane components. In this case, an $\mathbb{F}_q$-hyperplane of type (I) passes through some $Y \in B_2$ according to the proof of Lemma 2.10, while an $\mathbb{F}_q$-hyperplane of type (II) passes through some $Y \in B_1 \cup B_2 = B$.

On the other hand, every $Y \in B$ is contained in at most $(q + 1)$ hyperplanes defined over $\mathbb{F}_q$. Therefore, by Lemma 2.9, the members in $B_1$ are contained in at most

$$\left(\sum_{k=t+1}^{t} d_k(d - 1)(q + 1}\right) (q + 1) = (d - t)(d - 1)(q + 1)^2$$

hyperplanes over $\mathbb{F}_q$, and, by Lemma 2.10, the members in $B_2$ are contained in at most

$$\binom{t}{2}(q + 1) = \frac{1}{2}t(t - 1)(q + 1)$$

hyperplanes over $\mathbb{F}_q$. These contribute at most

$$(d - t)(d - 1)(q + 1)^2 + \frac{1}{2}t(t - 1)(q + 1).$$

hyperplanes of types (I) and (II).

Now assume that $t = 1$, that is, $X$ contains one and only one hyperplane component $X_1$. This forces $X_1$ to be defined over the ground field $\mathbb{F}_q$. In this case, the only hyperplane of type (I) is $X_1$, and it may or may not pass through a member of $B$. Therefore, we have to increase the previous bound by 1 for this case. □

Theorem 2.12. Let $X \subset \mathbb{P}^n$ be a reduced hypersurface of degree $d \geq 2$ over $\mathbb{F}_q$. Then there exists a hyperplane $H \subset \mathbb{P}^n$ over $\mathbb{F}_q$ such that $X \cap H$ is proper and reduced provided that
\[ q \geq d(d - 1) + 1 \text{ when } n = 3. \]
\[ q \geq d \text{ when } n \geq 4. \]

**Proof.** By Proposition 2.11, we will get a desirable hyperplane if
\[
\sum_{j=0}^{n} q^j > (d - t)(d - 1)(q + 1)^2 + \frac{1}{2}t(t - 1)(q + 1) + 1.
\]

We can cancel the constant 1 on the right side by starting the sum on the left with \( j = 1 \). In fact, we will ensure that a stronger inequality holds:

\[
\sum_{j=1}^{n} q^j > \left( (d - t)(d - 1) + \frac{1}{2}t(t - 1) \right) (q + 1)^2.
\]

We want to maximize the quantity
\[
\phi(t) := (d - t)(d - 1) + \frac{1}{2}t(t - 1)
\]
as a function of \( t \) on the interval \([0, d]\). Note that \( \phi(t) \) is a quadratic polynomial in \( t \) with leading term \((1/2)t^2\). As the graph of \( \phi(t) \) is the usual upward-facing parabola, the maximum is attained at the end point \( t = 0 \) or \( t = d \). Since \( \phi(0) = d(d - 1) \) and \( \phi(d) = \frac{1}{2}d(d - 1) \), we conclude that \( \phi(t) \leq d(d - 1) \).

Straightforward computations show that the inequality
\[
q^{n-3}(q - 1) \geq d(d - 1)
\]
holds in the following cases:

- \( n = 3 \) and \( q \geq d(d - 1) + 1 \),
- \( n \geq 4 \) and \( q \geq d \).

By hypothesis, we have \( n \geq 3 \), which implies that
\[
\sum_{j=1}^{n} q^j > q^{n-3}(q^3 + q^2 - q - 1) = q^{n-3}(q - 1)(q + 1)^2.
\]

Combining this with (2.3), we obtain
\[
\sum_{j=1}^{n} q^j > d(d - 1)(q + 1)^2 \geq \phi(t)(q + 1)^2
\]
\[
= \left( (d - t)(d - 1) + \frac{1}{2}t(t - 1) \right) (q + 1)^2.
\]

which is exactly (2.2), as desired. \[\square\]
Corollary 2.13. Let $X \subset \mathbb{P}^n$ be a reduced hypersurface of degree $d \geq 2$ over $\mathbb{F}_q$. Then, for every $2 \leq r \leq n - 1$, there exists an $r$-plane $T \subset \mathbb{P}^n$ over $\mathbb{F}_q$ such that $X \cap T$ is proper and reduced provided that

$$q \geq \frac{3}{2} d(d - 1).$$

Proof. Since $d \geq 2$, it is straightforward to verify that both inequalities in the hypothesis of Theorem 2.12 are satisfied, so there exists a hyperplane $H \subset \mathbb{P}^n$ over $\mathbb{F}_q$ such that $X \cap H$ is reduced with one dimension less than $X$. By repeating this process, we get hyperplanes $H = H_{n-1}, \ldots, H_2$ over $\mathbb{F}_q$ and a sequence of varieties

$$X = X_{n-1} \supset X_{n-2} \supset \cdots \supset X_1$$

such that $X_{i-1} = X_i \cap H_i$ is reduced for each $2 \leq i \leq n - 1$, and dim $X_j = j$ for each $1 \leq j \leq n - 1$. Note that, for $2 \leq r \leq n - 1$,

$$X_{r-1} = X \cap \bigcap_{j=1}^{n-r} H_{n-j}$$

is reduced and of expected dimension $r - 1$. Thus, $T := \cap_{j=1}^{n-r} H_{n-j} \cong \mathbb{P}^r$ is the desired $r$-plane over $\mathbb{F}_q$. □

Proof of Theorem 2.1. By Corollary 2.13, there exists a plane $H \cong \mathbb{P}^2$ in $\mathbb{P}^n$ over $\mathbb{F}_q$ such that $X_1 := X \cap H$ is a reduced plane curve. Now we apply Proposition 2.7 to find an $\mathbb{F}_q$-line $L \subset \mathbb{P}^2$ such that $X_1 \cap L$ consists of $d$ distinct points. This line $L$ also satisfies the condition that $(X \cap L) = d$ distinct points, and so $L$ is a desired transverse line to $X$. □

3. Transverse linear subspaces of arbitrary dimensions

Our goal in this section is to prove Theorem 1.3. More precisely, we will prove the following result.

Theorem 3.1. Let $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree $d \geq 3$ over $\mathbb{F}_q$. Assume $0 \leq r \leq n - 1$ and

$$q \geq (n-r)d(d-1)^r.$$ 

Then $X$ admits a very transverse $r$-plane defined over $\mathbb{F}_q$.

The notion of very transversality will be introduced in §3.1. By our definition, a very transverse linear subspace is automatically transverse, so Theorem 3.1 remains true for transverse $r$-planes. The proof of the theorem proceeds by induction on $r$, and the notion of very transversality is required in the inductive step.
3.1. **Very transversality and main strategy.** Let $X$ be a smooth hypersurface in $\mathbb{P}^n$. Recall that a linear subspace $H \subset \mathbb{P}^n$ of dimension $r$ is transverse to $X$ if $\dim(H \cap T_{P}X) = r - 1$ for every $P \in X \cap H$.

**Definition 3.2.** Let $X \subset \mathbb{P}^n$ be a smooth hypersurface. We say an $r$-plane $H \subset \mathbb{P}^n$ is *very transverse* to $X$ if

- it is transverse to $X$, and
- it is contained in a hyperplane that is transverse to $X$, or equivalently, $H^* \not\subset X^*$ in the dual space $(\mathbb{P}^n)^*$.

Let $\gamma: X \longrightarrow (\mathbb{P}^n)^*$ be the Gauss map. It is immediate that

$$
\gamma^{-1}(H^* \cap X^*) = \{ P \in X \mid T_{P}X \supset H \}.
$$

Since $X$ is a smooth hypersurface, its dual $X^*$ is a hypersurface, because the Gauss map is a finite morphism [Zak93, Corollary 2.8]. Under this condition, $H$ is very transverse if and only if it is transverse to $X$ and satisfies

$$
\dim(\gamma^{-1}(H^* \cap X^*)) = \dim(H^* \cap X^*) = n - r - 2
$$

**Example 3.3.** Let $X \subset \mathbb{P}^n$ be a smooth projective hypersurface and let $H \subset \mathbb{P}^n$ be a hyperplane. If $H$ is transverse to $X$, then $H^*$ is a point away from $X^*$, and thus $H$ is very transverse to $X$. In other words, a hyperplane is transverse if and only if it is very transverse.

**Example 3.4.** Suppose that $X \subset \mathbb{P}^n$ is a smooth hypersurface of degree $d \geq 3$ over $\mathbb{F}_q$. Then every $P \in \mathbb{P}^n(\mathbb{F}_q) \setminus X(\mathbb{F}_q)$ is a 0-plane very transverse to $X$. Indeed, $P$ is automatically transverse to $X$. Note that $X^*$ cannot be a hyperplane as it would imply that $X$ is *strange* (meaning that all tangent hyperplanes pass through a common point), but the only non-linear smooth strange hypersurfaces are quadrics of odd dimensions in characteristic 2 [KP91, Theorem 7]. As $P^*$ is a hyperplane and $X^*$ is irreducible, the condition $P^* \not\subset X^*$ holds.

**Example 3.5.** Let $V$ be a vector space of dimension $n + 1$ over a field $k$ with $\text{char}(k) \neq 2$ and let $b$ be a nondegenerate bilinear form on $V$. In particular, $b$ induces an isomorphism

$$
(3.1) \quad V \sim \rightarrow V^* : x \mapsto b(x, \cdot).
$$

For every subspace $W \subset V$ where $b|_W$ is nondegenerate, there is an orthogonal decomposition

$$
V = W \oplus W^\perp
$$

such that $b|_{W^\perp}$ is nondegenerate. Let us translate this fact using projective geometry: The form $b$ defines a smooth quadric $X \subset \mathbb{P}^n \cong \mathbb{P}(V)$,
and the Gauss map

\[ \gamma: X \sim \rightarrow X^* \subset (\mathbb{P}^n)^* \]

is an isomorphism since it is compatible with (3.1). Under this identification, the fact above is equivalent to saying that a linear subspace \( H = \mathbb{P}(W) \) is transverse to \( X \) if and only if its dual \( H^* = \mathbb{P}(W^\perp) \) is transverse to \( X^* \). Therefore, the notions of transversality and very transversality coincide for smooth quadrics.

**Remark 3.6.** In characteristic zero, transversality coincides with very transversality due to the general version of Bertini’s theorem (see, for example, [Kle74, Corollary 5]). We do not know if these two notions coincide in positive characteristics.

The proof strategy for Theorem 3.1 will be by induction on \( r \). Suppose that there exists a very transverse \((r - 1)\)-plane \( H_{r-1} \) defined over \( \mathbb{F}_q \). In order to find a very transverse \( r \)-plane \( H_r \), we will consider all the \( r \)-planes defined over \( \mathbb{F}_q \) containing \( H_{r-1} \), and show that at least one of them is very transverse to \( X \).

In order to achieve this, we need to estimate the number of bad choices, that is, the number of \( r \)-planes containing \( H_{r-1} \) which are not very transverse to \( X \). By Definition 3.2, such an \( r \)-plane \( H \) is

- not transverse to \( X \), or
- satisfies \( H^* \subset X^* \).

Let us estimate the second kind of \( r \)-planes below, and leave the estimate for the first kind of \( r \)-planes to the next subsection.

**Proposition 3.7.** Let \( X \subset \mathbb{P}^n \) be a smooth hypersurface of degree \( d \) over \( \mathbb{F}_q \) and let \( H_{r-1} \) be a very transverse \((r - 1)\)-plane to \( X \). Then the number of \( r \)-planes \( H \) over \( \mathbb{F}_q \) which contains \( H_{r-1} \) and satisfies \( H^* \subset X^* \) is at most \( d(d - 1)^{n-1} \).

**Proof.** Since \( H_{r-1} \) is very transverse, the intersection \( H_{r-1}^* \cap X^* \) is a hypersurface in \( H_{r-1}^* \). Note that the \( r \)-planes \( H \supset H_{r-1} \) is in one-to-one correspondence with the hyperplanes \( H^* \subset H_{r-1}^* \), and \( H \) is over \( \mathbb{F}_q \) if and only if \( H^* \) is over \( \mathbb{F}_q \). Hence the \( r \)-planes \( H \supset H_{r-1} \) over \( \mathbb{F}_q \) that satisfy \( H^* \subset X^* \) correspond to the hyperplane components of \( H_{r-1}^* \cap X^* \) defined over \( \mathbb{F}_q \). The number of such components is bounded by the degree,

\[ \deg(H_{r-1}^* \cap X^*) = \deg(X^*) \leq d(d - 1)^{n-1}, \]

from which the conclusion follows. \( \square \)
3.2. Estimate for the number of tangent subspaces. In the following, we fix a smooth hypersurface

\[ X = \{ F = 0 \} \subset \mathbb{P}^n \]

over \( \mathbb{F}_q \), an integer \( 1 \leq r \leq n - 1 \), and an \((r - 1)\)-plane \( H_{r-1} \subset \mathbb{P}^n \) over \( \mathbb{F}_q \) which is very transverse to \( X \). In this section, we estimate the number of \( r \)-planes containing \( H_{r-1} \) that are tangent to \( X \). Combining the result with Proposition 3.7 will give us an upper bound for the number of \( r \)-planes containing \( H_{r-1} \) that are not very transverse to \( X \).

Upon possibly applying a \( \text{PGL}_{n+1}(\mathbb{F}_q) \)-action, let us fix a coordinate system \( \{ x_0, \ldots, x_n \} \) for \( \mathbb{P}^n \) such that \( H_{r-1} = \{ x_r = \cdots = x_n = 0 \} \).

For the sake of simplicity, we denote \( F_i := \partial F / \partial x_i \). Recall that the Gauss map associated with \( X \) is given by

\[ \gamma : X \to X^* : [x_0 : \cdots : x_n] \mapsto [F_0 : \cdots : F_n]. \]

For each \( r \in \{ 0, \ldots, n \} \), we define

\[ X_{n-r-1} := \gamma^{-1}(H_{r-1}^* \cap X^*) = X \cap \bigcap_{i=0}^{r-1} \{ F_i = 0 \}. \]

The hypothesis that \( H_{r-1} \) is very transverse to \( X \) implies that

\[ \dim(H_{r-1}^* \cap X^*) = \dim(X^*) - r = n - r - 1 \]

By Zak’s theorem on the finiteness of \( \gamma \), the preimage \( X_{n-r-1} \) also has dimension \( n - r - 1 \). In particular, \( X_{n-r-1} \) is a complete intersection.

Now consider the \((r + 2)\)-by-\((n + 1)\) matrix

\[ M_r := \begin{pmatrix}
1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\
x_0 & x_1 & \cdots & x_{r-1} & x_r & \cdots & x_n \\
x_0^q & x_1^q & \cdots & x_{r-1}^q & x_r^q & \cdots & x_n^q
\end{pmatrix} \]

which consists of an \( r \)-by-\( r \) identity minor on the top left, the coordinates \( (x_0, x_1, \ldots, x_n) \) and their \( q \)-th power in the last two rows, and zeros on the remaining entries. Notice that \( H_{r-1} \) is spanned by the first \( r \) rows of \( M_r \). Let us define

\[ Z_r \subset \mathbb{P}^n \]

to be the determinantal variety of the maximal minors of \( M_r \). Note that \( Z_r \) is the union of \( r \)-planes over \( \mathbb{F}_q \) containing \( H_{r-1} \).
Let us label the columns of $M_r$ from 0 to $n$, and for $r \leq i < j \leq n$, let us denote by

$$V_{ij} \subset \mathbb{P}^n$$

the hypersurface defined by the maximal minor consisting of the first $r$ columns and the $i$ and $j$-th columns. It is straightforward to see that

$$V_{ij} = \{ x_i^q x_j - x_i x_j^q = 0 \}$$

and also that

$$Z_r = \bigcap_{r \leq i < j \leq n} V_{ij}.$$  

Note that each $V_{ij}$ is completely reducible over $\mathbb{F}_q$. In fact, $V_{ij}$ is the union of $q + 1$ hyperplanes \( \{ ax_i + bx_j = 0 \} \) as \( [a : b] \) varies in \( \mathbb{P}^1(\mathbb{F}_q) \).

**Definition 3.8.** Given a closed subscheme $W \subset \mathbb{P}^n$, write $W = \bigcup_k W_k$ as the union of subschemes $W_k$ of pure dimension $k$. We define the mixed degree of $W$, and still write as $\deg(W)$, to be $\sum_k \deg(W_k)$.

**Lemma 3.9.** Let $Y \subset \mathbb{P}^n$ be a subscheme of pure dimension $m$ defined over $\mathbb{F}_q$. Then $\deg(Y \cap Z_r) \leq \deg(Y) \cdot (q + 1)^m$.

**Proof.** We proceed by induction on $m$. When $m = 0$, we have

$$\deg(Y \cap Z_r) \leq \deg(Y) = \deg(Y) \cdot (q + 1)^0$$

so the statement holds. For the inductive step, suppose that the statement is true in all dimensions less than $m$. Given $Y$ with dimension $m$, we decompose

$$Y = Y_1 \cup \cdots \cup Y_\ell$$

into $\mathbb{F}_q$-irreducible components.

For every $k \in \{1, \ldots, \ell\}$, let us show that

$$\deg(Y_k \cap Z_r) \leq \deg(Y_k) \cdot (q + 1)^m. \quad (3.2)$$

If there exists $V_{ij}$ such that $Y_k' := Y_k \cap V_{ij}$ has dimension $m - 1$, then, by induction,

$$\deg(Y_k \cap Z_r) = \deg(Y_k \cap V_{ij} \cap Z_r) = \deg(Y_k' \cap Z_r) \leq \deg(Y_k') \cdot (q + 1)^{m-1} = \deg(Y_k) \cdot (q + 1)^m.$$ 

If $Y_k \cap V_{ij}$ has dimension $m$ for all $r \leq i < j \leq n$, then $Y_k \subset V_{ij}$ for all $i$ and $j$ as $Y_k$ is $\mathbb{F}_q$-irreducible. This implies that $Y_k \subset Z_r$ and thus

$$\deg(Y_k \cap Z_r) = \deg(Y_k) \leq \deg(Y_k)(q + 1)^m.$$ 

Hence (3.2) holds. It follows that

$$\deg(Y \cap Z_r) \leq \sum_{k=1}^\ell \deg(Y_k \cap Z_r)$$
\[
\leq \sum_{k=1}^{\ell} \operatorname{deg}(Y_k) \cdot (q + 1)^m = \operatorname{deg}(Y) \cdot (q + 1)^m.
\]

as desired. \qed

Lemma 3.10. Let \( X \subset \mathbb{P}^n \) be a smooth hypersurface over \( \mathbb{F}_q \) and let \( H_{r-1} \) be a very transverse \((r-1)\)-plane to \( X \). Then each \( \mathbb{F}_q \)-component of \( X_{n-r-1} \cap Z_r \) is contained in an \( r \)-plane containing \( H_{r-1} \) that is defined over \( \mathbb{F}_q \) and tangent to \( X \). Conversely, every such \( r \)-plane contains an \( \mathbb{F}_q \)-component of \( X_{n-r-1} \cap Z_r \).

Proof. By construction, each \( \mathbb{F}_q \)-component of \( X_{n-r-1} \cap Z_r \) is contained in an \( r \)-plane \( E \) containing \( H_{r-1} \) defined over \( \mathbb{F}_q \). To prove that \( E \) is tangent to \( X \), let \( P \in X_{n-r-1} \cap E \). The condition \( P \in X_{n-r-1} \) implies that \( \gamma(P) \in H^*_{r-1} \) and thus \( T_P X \supset H_{r-1} \). The last property implies that \( P \not\in H_{r-1} \) since \( H_{r-1} \) is transverse to \( X \). We conclude that

\[ T_P X \supset \langle P, H_{r-1} \rangle = E \ni P \]

and hence \( E \) is tangent to \( X \).

In order to prove the converse, let us consider the set

\[ T := \{ E \mid E \text{ is an } r \text{-plane defined over } \mathbb{F}_q \text{ such that } E \text{ contains } H_{r-1} \text{ and } E \text{ is tangent to } X \} \]

(3.3)

We claim that

1. If \( E \in T \), then \( X_{n-r-1} \cap E \neq \emptyset \).
2. If \( E_1, E_2 \) are two distinct elements of \( T \), then \( X_{n-r-1} \cap E_1 \) and \( X_{n-r-1} \cap E_2 \) are disjoint.

To prove (1), note that \( E \in T \) implies that \( E \subset T_P X \) for some \( P \in E \). The tangent hyperplane \( T_P X \) contains \( E \) and thus contains \( H_{r-1} \), so \( P \) belongs to \( X_{n-r-1} \) by definition. Hence \( P \in X_{n-r-1} \cap E \) and thus \( X_{n-r-1} \cap E \neq \emptyset \). To prove (2), assume there exists a common point

\[ P \in X_{n-r-1} \cap E_1 \cap E_2. \]

The condition \( P \in X_{n-r-1} \) implies that \( T_P X \supset H_{r-1} \) and thus \( P \not\in H_{r-1} \) since \( H_{r-1} \) is transverse to \( X \). Therefore, we have

\[ E_1 = \langle P, H_{r-1} \rangle = E_2 \]

which proves the claim.

Property (1) implies that the intersection

\[ (X_{n-r-1} \cap Z_r) \cap E = X_{n-r-1} \cap E \]

is nonempty and thus is contained in an \( \mathbb{F}_q \)-component \( Y \) of \( X_{n-r-1} \cap Z_r \).

From what we have proved before, there exists \( E' \in T \) such that \( Y \subset \)
Then $E = E'$ and thus $E \supset Y$. This completes the proof. \qed

**Remark 3.11.** Let $S$ be the set of $\mathbb{F}_q$-components of $X_{n-r-1} \cap Z_r$ and let $T$ be the same as (3.3). Lemma 3.10 defines a surjective map $S \rightarrow T$. In particular, we have $|S| \geq |T|$.

**Proposition 3.12.** Let $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree $d$ over $\mathbb{F}_q$ and let $H_{r-1}$ be a very transverse $(r-1)$-plane to $X$. Then the number of $r$-planes containing $H_{r-1}$ defined over $\mathbb{F}_q$ which are tangent to $X$ is at most $d(d-1)^r(q+1)^{n-r-1}$.

**Proof.** By Lemma 3.10 (see also Remark 3.11), the number of $r$-planes containing $H_{r-1}$ which are defined over $\mathbb{F}_q$ and also tangent to $X$ is bounded by the number of $\mathbb{F}_q$-irreducible components of $X_{n-r-1} \cap Z_r$, which is bounded by its own (mixed) degree. To estimate the latter, we apply Lemma 3.9 to obtain

$$\deg(X_{n-r-1} \cap Z_r) \leq \deg(X_{n-r-1})(q+1)^{\dim X_{n-r-1}} = d(d-1)^r(q+1)^{n-r-1}.$$  

This completes the proof. \qed

**Corollary 3.13.** Let $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree $d$ over $\mathbb{F}_q$ and let $H_{r-1}$ be a very transverse $(r-1)$-plane to $X$. Then the number of $r$-planes containing $H_{r-1}$ over $\mathbb{F}_q$ which are not very transverse to $X$ is at most $d(d-1)^r(q+1)^{n-r-1} + d(d-1)^{n-1}$.

**Proof.** This follows from Propositions 3.7 and 3.12. \qed

**3.3. Existence of very transverse linear subspaces.** In this section, we finish the proof of Theorem 3.1. We will proceed by induction on $r$ and divide the proof into a number of lemmas with respect to the following cases:

- $r = 0$, the initial case;
- $1 \leq r \leq n-3$;
- $r = n-2$;
- $r = n-1$.

To prove the initial case, we need the following result:

**Proposition 3.14.** Let $X \subset \mathbb{P}^n$ be a hypersurface over $\mathbb{F}_q$ such that $X(\mathbb{F}_q) = \mathbb{P}^n(\mathbb{F}_q)$. Then $\deg(X) \geq q+1$.

**Proof.** We proceed by induction on $n$. When $n = 1$, the conclusion follows since a space-filling subset $X \subset \mathbb{P}^1$ must be defined by a binary form which is divisible by $x^qy - xy^q$. For the inductive step, suppose
that $X \subset \mathbb{P}^n$ is a hypersurface with $X(\mathbb{F}_q) = \mathbb{P}^n(\mathbb{F}_q)$ where $n \geq 2$. If $X$ contains all $\mathbb{F}_q$-hyperplanes $T \subset \mathbb{P}^n$, then
\[
\deg(X) \geq q^n + \cdots + q + 1 > q + 1.
\]
Otherwise, there exists a hyperplane $T \subset \mathbb{P}^n$ defined over $\mathbb{F}_q$ such that $\dim(X \cap T) = \dim(X) - 1$. Now, $X \cap T \subset T \cong \mathbb{P}^{n-1}$ can be viewed as a hypersurface in a projective space of one lower dimension, and satisfies $(X \cap T)(\mathbb{F}_q) = \mathbb{P}^{n-1}(\mathbb{F}_q)$. Applying the inductive hypothesis to $X \cap T$, we obtain $\deg(X) = \deg(X \cap T) \geq q + 1$. □

By Example 3.4, the initial case $r = 0$ is equivalent to the following:

**Lemma 3.15.** Let $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree $d \geq 3$ over $\mathbb{F}_q$. Assume that $q \geq nd$. Then there exists $P \in \mathbb{P}^n(\mathbb{F}_q)$ which is not on $X$.

*Proof.* By hypothesis, we have $q \geq nd \geq d > d - 1$. Thus $d < q + 1$, which means that $X$ cannot be space-filling, i.e. $X(\mathbb{F}_q) \neq \mathbb{P}^n(\mathbb{F}_q)$. Indeed, the minimum degree of a space-filling hypersurface in $\mathbb{P}^n$ is $q + 1$ by Proposition 3.14. □

Let us prove a lemma before entering the inductive step:

**Lemma 3.16.** Suppose that $d \geq 3$ and $1 \leq r \leq n - 2$. Then
\[
q \geq (n - r)d(d - 1)^r \quad \text{implies} \quad q^{n-r} > d(d - 1)^r(q + 1)^{n-r-1}.
\]

*Proof.* The claimed inequality is equivalent to
\[
(3.4) \quad \left(\frac{q}{q + 1}\right)^{n-r-1} \cdot q > d(d - 1)^r.
\]

First, we claim that
\[
(3.5) \quad \left(1 - \frac{1}{6m}\right)^m > \frac{1}{m+1}
\]
for any integer $m \geq 1$. Using Bernoulli’s inequality [MP93], which states that $(1 + x)\ell \geq 1 + \ell x$ for every integer $\ell \geq 0$ and every real number $x \geq -1$, we obtain
\[
\left(1 - \frac{1}{6m}\right)^m \geq 1 - \left(\frac{1}{6m}\right)^m \geq \frac{5}{6} \geq \frac{1}{2} \geq \frac{1}{m+1}.
\]
This proves (3.5). Since $d \geq 3$ and $r \geq 1$, we have
\[
q + 1 > q \geq (n - r)d(d - 1)^r \geq 6(n - r) > 6(n - r - 1).
\]
It follows that
\[
\left(\frac{q}{q + 1}\right)^{n-r-1} \cdot q = \left(1 - \frac{1}{q+1}\right)^{n-r-1} \cdot q
\]

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\[ \geq \left( 1 - \frac{1}{6(n-r-1)} \right)^{n-r-1} \cdot q \]
\[ \geq \left( 1 - \frac{1}{6(n-r-1)} \right)^{n-r-1} \cdot (n-r)d(d-1)^r \]

(using (3.5))
\[ > \frac{1}{n-r} \cdot (n-r)d(d-1)^r = d(d-1)^r \]

which proves the claimed inequality (3.4). \ \Box

For the inductive step to work properly, we need to check that:

**Lemma 3.17.** Suppose that \( n > r \) and \( d \geq 3 \). Then the hypothesis
\[ q \geq (n-r)d(d-1)^r \]
for step \( r \) in the inductive step implies
\[ q \geq (n-(r-1))d(d-1)^{r-1} \]
for step \( r-1 \).

**Proof.** Using the assumption that \( n > r \) and \( d \geq 3 \), we get
\[ d - 1 \geq 1 + \frac{1}{n-r} \]
which is equivalent to
\[ (n-r)(d-1) \geq n - r + 1. \]
It follows that
\[ q \geq (n-r)d(d-1)^r = (n-r)(d-1) \cdot d(d-1)^{r-1} \]
\[ \geq (n-r+1)d(d-1)^{r-1} \]
as desired. \ \Box

In the inductive step, Lemma 3.17 assures the existence of an \((r-1)\)-plane \( H_{r-1} \) over \( \mathbb{F}_q \) which is very transverse to \( X \). By Corollary 3.13, the number of \( r \)-planes defined over \( \mathbb{F}_q \) containing \( H_{r-1} \) which are not very transverse to \( X \) is bounded above by
\[ d(d-1)^r(q + 1)^{n-r-1} + d(d-1)^{n-1} \]
On the other hand, the total number of \( r \)-planes over \( \mathbb{F}_q \) containing \( H_{r-1} \) equals \( \sum_{i=0}^{n-r} q^i \). Therefore, we obtain an \( r \)-plane over \( \mathbb{F}_q \) that is very transverse to \( X \) provided that
\[ \sum_{i=0}^{n-r} q^i > d(d-1)^r(q + 1)^{n-r-1} + d(d-1)^{n-1} \]
We claim that this inequality is satisfied under the hypothesis
\[ q \geq (n - r)d(d - 1)^r \]
of the theorem for \( 1 \leq r \leq n - 2 \). We will divide the proof into two cases. The situation \( r = n - 1 \) will be treated later.

**Lemma 3.18.** Suppose that \( 1 \leq r \leq n - 3 \). Then inequality (3.6) holds provided that \( q \geq (n - r)d(d - 1)^r \).

**Proof.** In this case, (3.6) will follow from the two inequalities:
\[
q^{n-r} > d(d - 1)^r(q + 1)^{n-r-1} \\
q^{n-r-1} > d(d - 1)^{n-1} 
\]
The first inequality is proved in Lemma 3.16. As for the second one, using the hypothesis \( q \geq (n - r)d(d - 1)^r > d(d - 1)^r \), we have,
\[
q^{n-r-1} > d^{n-r-1}(d - 1)^r(n-r-1) \\
> d(d - 1)^{n-r-2}(d - 1)^r(n-r-1) \\
= d(d - 1)^{n-r-2+r(n-r-1)} \\
= d(d - 1)^{n-1+r(n-r-1)-r-1} \\
\text{(as } n \geq r + 3) \\
\geq d(d - 1)^{n-1+r(r+3-r-1)-r-1} \\
= d(d - 1)^{n-1+r-1} \\
\text{(as } r \geq 1) \\
\geq d(d - 1)^{n-1} 
\]
as desired. \( \square \)

**Lemma 3.19.** Suppose that \( r = n - 2 \). Then inequality (3.6) holds provided that \( q \geq (n - r)d(d - 1)^r \).

**Proof.** When \( r = n - 2 \), our hypothesis is equivalent to \( q \geq 2d(d - 1)^{n-2} \), and inequality (3.6) reduces to:
\[
q^2 + q + 1 > d(d - 1)^{n-2}(q + 1) + d(d - 1)^{n-1} \\
= d(d - 1)^{n-2}(q + d) 
\]
We prove this last inequality as follows:
\[
q^2 + q + 1 > q(q + 1) \\
\text{(as } q \geq 2d(d - 1)^{n-2}) \\
\geq 2d(d - 1)^{n-2}(q + 1) \\
> 2d(d - 1)^{n-2} \cdot q \\
\geq d(d - 1)^{n-2}(q + d) 
\]
This justifies (3.6) for the case \( r = n - 2 \). \( \square \)
Proof of Theorem 3.1. The proof proceeds by induction on \( r \). The initial step \( r = 0 \) is done in Lemma 3.15. The inductive step is built upon Lemma 3.17 and the cases \( 1 \leq r \leq n - 2 \) are accomplished by proving (3.6) in Lemmas 3.18 and 3.19.

For the remaining case \( r = n - 1 \), it is enough to prove the existence of a transverse \( \mathbb{F}_q \)-hyperplane since a transverse hyperplane is necessarily very transverse by Example 3.3. Thus, we only need to take into account the contributions coming from tangent hyperplanes. By Proposition 3.12 applied with \( r = n - 1 \), there are at most \( d(d - 1)^{n-1} \) tangent hyperplanes over \( \mathbb{F}_q \) containing \( H_{n-2} \). Since there are \( q + 1 \) hyperplanes over \( \mathbb{F}_q \) containing \( H_{n-2} \), it suffices to show that

\[
q + 1 > d(d - 1)^{n-1}
\]

which follows immediately from our hypothesis \( q \geq d(d - 1)^{n-1} \). \( \square \)

References


S. Asgarli, Department of Mathematics
University of British Columbia
Vancouver, BC V6T1Z2, Canada
sasgarli@math.ubc.ca