Exercises.

1. Recall, from class, that for a pair of relatively prime positive integers $p$ and $q$ we can form a knot with parametrization

$$
\begin{align*}
  x(t) &= (2 + \cos(pt)) \cos(qt) \\
  y(t) &= (2 + \cos(pt)) \sin(qt) \\
  z(t) &= \sin(pt)
\end{align*}
$$

where $0 \leq t \leq 2\pi$.

(a) Consider the knot obtained in the case $p = 2, q = 3$, and consider the result of projecting this to the $xy$-plane and to the $yz$-plane. Do both these projections give rise to knot diagrams? Why or why not?

(b) Find, with proof, values of $p$ and $q$ for which this parametrization gives the knot $5_1$.

2. Figure 1.24 in §1.3 of Adams depicts the following two distinct Reidemeister moves of type III:

   (1)
   \[
   \begin{array}{c}
   \text{X} \\
   \text{X}
   \end{array}
   \quad \leftrightarrow 
   \quad
   \begin{array}{c}
   \text{X} \\
   \text{X}
   \end{array}
   
   \]

(2)

Using the Reidemeister moves of type I and II and the move (1), deduce the move (2). (It is also possible to show that the move (1) follows from the type I and II moves together with (2)).

3. Links are introduced in Adams §1.4. Show that the following diagrams describe the same link:

   In your solution, explicitly identify where you made use of a Reidemeister Type III move.

4. Decide the 3-colourability of each of the knots $7_{5}$, $7_{6}$, $7_{7}$: In each case, either give a valid 3-colouring or prove that no 3-colouring exists.
Problems

5. The following infinite family of knots are referred to as twist knots:

(a) Check that the case \( n = 1 \) is the knot \( 3_1 \); and that the case \( n = 4 \) is the knot \( 6_1 \).

(b) Prove that a twist knot is 3-colourable if and only if \( n = 3k + 1 \). (Recall that an if and only if statement requires that you check two implications!)

6. Using the notation from the table in Adams, the figure eight knot is the knot \( 4_1 \) and the trefoil knot is the knot \( 3_1 \).

(a) A 5-colouring of a knot diagram is a choice of label from the set \( \{0, 1, 2, 3, 4\} \) (these are the five “colours”) for each of the strands so that at least two colours are used and at each crossing, if \( z \) is the label on the overstrand, the equation

\[ x + y - 2z = 0 \text{ modulo } 5 \]

holds. Show that the existence/non-existence of such a colouring is preserved under each of the Reidemeister moves.

(b) Show that the figure eight knot is 5-colourable.

(c) Using material that we have seen in the course and homework to this point, prove that the unknot, the trefoil, and the figure eight knot are mutually distinct.