INFINITE TIME BLOW-UP IN THE KELLER-SEGEL SYSTEM: EXISTENCE AND STABILITY

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ABSTRACT. Perhaps the most classical diffusion model for chemotaxis is the Keller-Segel system
\begin{align*}
\begin{cases}
    u_t &= \Delta u - \nabla \cdot (u \nabla v) \quad \text{in } \mathbb{R}^2 \times (0, \infty), \\
    v &= (-\Delta_{\mathbb{R}^2})^{-1} u := \frac{1}{2\pi} \int_{\mathbb{R}^2} \log \frac{1}{|x-z|} u(z, t) \, dz \\
    u(\cdot, 0) &= u_0 \geq 0 \quad \text{in } \mathbb{R}^2.
\end{cases}
\end{align*}
(0.1)

We consider the critical mass case \( \int_{\mathbb{R}^2} u_0(x) \, dx = 8\pi \) which corresponds to the exact threshold between finite-time blow-up and self-similar diffusion towards zero. We find a radial function \( u^*_0 \) with mass \( 8\pi \) such that for any initial condition \( u_0 \) sufficiently close to \( u^*_0 \) the solution \( u(x,t) \) of (0.1) is globally defined and blows-up in infinite time. As \( t \to +\infty \) it has the approximate profile
\begin{align*}
    u(x,t) &\approx \frac{1}{\lambda(t)} U_0 \left( \frac{x - \xi(t)}{\lambda(t)} \right), \\
    U_0(y) &= \frac{8}{(1 + |y|^2)^2},
\end{align*}
where \( \lambda(t) \approx c \sqrt{\log t} \), \( \xi(t) \to q \) for some \( c > 0 \) and \( q \in \mathbb{R}^2 \).

1. INTRODUCTION

This paper deals with the classical Keller-Segel problem in \( \mathbb{R}^2 \),
\begin{align*}
\begin{cases}
    u_t &= \Delta u - \nabla \cdot (u \nabla v) \quad \text{in } \mathbb{R}^2 \times (0, \infty), \\
    v &= (-\Delta_{\mathbb{R}^2})^{-1} u := \frac{1}{2\pi} \int_{\mathbb{R}^2} \log \frac{1}{|x-z|} u(z, t) \, dz \\
    u(\cdot, 0) &= u_0 \quad \text{in } \mathbb{R}^2
\end{cases}
\end{align*}
(1.1)

which is a well-known model for the dynamics of a population density \( u(x,t) \) evolving by diffusion with a drift representing a chemotaxis effect [16]. We consider positive solutions which are well defined and unique and smooth up to a maximal \( 0 < T \leq +\infty \). This problem formally preserves mass, in the sense that
\[ \int_{\mathbb{R}^2} u(x,t) \, dx = \int_{\mathbb{R}^2} u_0(x) \, dx =: M \quad \text{for all } \ t \in (0, T). \]

An interesting feature of (1.1) is the connection between the second moment of the solution and its mass which is precisely given by
\[ \frac{d}{dt} \int_{\mathbb{R}^2} u(x,t) |x|^2 \, dx = 4M - \frac{M^2}{2\pi}, \]
provided that the second moments are finite. If \( M > 8\pi \), the negative rate of production of the second moment and the positivity of the solution implies finite blow-up time. If \( M < 8\pi \) the solution lives at all times and eventually diffuses to

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zero with a self similar profile \[3\]. When \( M = 8\pi \) the solution is globally defined in time \[4\]. If the initial second moment is finite, it is preserved in time, and there is infinite time blow-up for the solution \[4\].

Globally defined in time solutions of \( (1.1) \) are of course its positive finite mass steady states, which consist of the family

\[
U_{\lambda, \xi}(x) = \frac{1}{\lambda} U_0 \left( \frac{x - \xi}{\lambda} \right), \quad U_0(y) = \frac{8}{(1 + |y|^2)^2}, \quad \lambda > 0, \, \xi \in \mathbb{R}^2. \tag{1.2}
\]

We observe that all these steady states have the exact mass \( 8\pi \) and infinite second moment

\[
\int_{\mathbb{R}^2} U_{\lambda, \xi}(x) \, dx = 8\pi, \quad \int_{\mathbb{R}^2} |x|^2 U_{\lambda, \xi}(x) \, dx = +\infty.
\]

In \[1, 14, 18\] blow-up in a bounded domain is studied, see also \[15\] and \[19\] for the higher-dimensional radial case. Asymptotics were derived in \[20, 21\] and a rigorous construction of a radial blow-up solution was found in \[17\]. Asymptotic stability of family of steady states \( (1.2) \) under this mass constraint has been determined in \[12\].

In the critical mass \( M = 8\pi \) case, the infinite-time blow-up in \( (1.1) \) when the second moment is finite, takes place in the form of a bubble in the form \( (1.2) \) with \( \lambda = \lambda(t) \to 0 \) \[4, 2\]. Formal rates and precise profiles were derived in \[6, 5\] to be \( \lambda(t) \sim \frac{c}{\sqrt{\log t}} \) as \( t \to +\infty \). A radial solution with this rate was built in \[13\] and its stability within the radial class was established. The stability assertion for general small perturbations was conjectured and left open in \[13\]. The method of construction in \[13\] seems difficult to adapt to the general, nonradial scenario.

In this paper we construct an infinite-time blow-up solution with an entirely different method to that in \[13\], which in particular leads to a proof of the stability assertion. The following is our main result.

**Theorem 1.** There exists a nonnegative function \( u_0^*(x) \) (radially symmetric) with critical mass \( \int_{\mathbb{R}^2} u_0^*(x) \, dx = 8\pi \) and finite second moment \( \int_{\mathbb{R}^2} u_0^*(x) \, dx < +\infty \) such that for every \( u_0(x) \) sufficiently close (in suitable sense) to \( u_0^* \) with \( \int_{\mathbb{R}^2} u_0 = 8\pi \) we have that the associated solution \( u(x,t) \) of system \( (1.1) \) has the form

\[
u(x,t) = \frac{1}{\lambda(t)^2} U_0 \left( \frac{x - \xi(t)}{\lambda(t)} \right) (1 + o(1)), \quad U_0(y) = \frac{8}{(1 + |y|^2)^2}
\]

uniformly on bounded sets of \( \mathbb{R}^n \), and

\[
\lambda(t) = \frac{c}{\sqrt{\log t}} (1 + o(1)), \quad \xi(t) \to q \quad \text{as} \quad t \to +\infty,
\]

for some number \( c > 0 \) and \( q \in \mathbb{R}^2 \).

Sufficiently close for the perturbation \( u_0(x) := u_0^*(x) + \varphi(x) \) in this result is measured in the \( C^1 \)-weighted norm for some \( \sigma > 0 \)

\[
\|\varphi\|_{C^1} := \|(1 + |\cdot|^{1+\sigma})\varphi\|_{L^\infty(\mathbb{R}^2)} + \|(1 + |\cdot|^{3+\sigma})\nabla \varphi(x)\|_{L^\infty(\mathbb{R}^2)} < +\infty. \tag{1.3}
\]

We observe that for any \( \sigma > 0 \) this decay condition implies that the second moment of \( \varphi \) is finite, which is not the case for \( \sigma \leq 0 \).

We devote the rest of this paper to the proof of Theorem 1. Our approach borrows elements of constructions in the works \[7, 9, 10, 11\] based on the so-called *inner-outer gluing scheme*, where a system is derived for an inner equation defined
near the blow-up point and expressed in the variable of the blowing-up bubble, and an outer problem that sees the whole picture in the original scale. The result of Theorem 1 has already been announced in \[8\] in connection with \[7, 9, 10\].

The scaling parameter is rather simple to find at main order from the approximate conservation of second moment. The center \(\xi(t)\) actually obeys a relatively simple system of nonlocal ODEs.

2. The ansatz

We consider the Keller-Segel system in the whole \(\mathbb{R}^2\)

\[
\begin{aligned}
    u_t &= \Delta u - \nabla \cdot (u \nabla v) \quad \text{in } \mathbb{R}^2 \times (0, \infty), \\
    v &= (-\Delta)^{-1} u := \frac{1}{2\pi} \int_{\mathbb{R}^2} \log \frac{1}{|x - z|} u(z, t) \, dz \\
    u(\cdot, 0) &= u_0 \quad \text{in } \mathbb{R}^2.
\end{aligned}
\] (2.1)

We will build a first approximation to a solution \(u(x, t)\) globally defined in time such that on bounded sets in \(x\) we have

\[
u(x, t) = \frac{1}{\lambda(t)^2} U_0 \left( \frac{x - \xi(t)}{\lambda(t)} \right) (1 + o(1)) \quad \text{as } t \to +\infty
\] (2.2)

for certain functions \(0 < \lambda(t) \to 0\) and \(\xi(t) \to q \in \mathbb{R}^2\). Here we recall

\[
U_0(y) = \frac{8}{(1 + |y|^2)^2}.
\]

2.1. Formal derivation of \(\lambda(t)\). We know that (2.2) can only happen in the critical mass, finite second moment case:

\[
\int_{\mathbb{R}^2} u(x, t) \, dx = 8\pi, \quad \int_{\mathbb{R}^2} |x|^2 u(x, t) \, dx < +\infty,
\]

which according to the results in \[4, 13, 6\] is consistent with a behavior of the form (2.2). Since the second moment of \(U_0\) is infinite, we do not expect the approximation (2.2) be uniform in \(\mathbb{R}^2\) but sufficiently far, a faster decay in \(x\) should take place.

We will find an approximate asymptotic expression for the scaling parameter \(\lambda(t)\) that matches with this behavior.

Let us introduce the function \(V_0 := (-\Delta)^{-1} U_0\). We directly compute

\[
V_0(y) = \log \frac{8}{(1 + |y|^2)^2}
\]

and hence \(V_0\) solves Liouville equation

\[-\Delta V_0 = e^{V_0} = U_0 \quad \text{in } \mathbb{R}^2.
\]

Then \(\nabla V_0(y) \approx -\frac{4y}{|y|^2}\) for all large \(y\), and hence we get, away from \(x = \xi\),

\[-\nabla \cdot (u \nabla (-\Delta)^{-1} u) \approx 4\nabla u \cdot \frac{x - \xi}{|x - \xi|^2}.
\]

Hence defining

\[
E(u) := \Delta u - \nabla \cdot (u \nabla (-\Delta)^{-1} u)
\] (2.3)

and writing in polar coordinates

\[
u(r, \theta, t) = u(x, t), \quad x = \xi(t) + re^{i\theta},
\]
we find $E(u) \approx \partial_t^2 u + \frac{5}{r} \partial_r u$ and hence, assuming $\dot{\xi}(t) \to 0$ sufficiently fast, equation (2.1) approximately reads

$$\partial_t u = \partial_t^2 u + \frac{5}{r} \partial_r u,$$

which can be idealized as a homogeneous heat equation in $\mathbb{R}^6$ for radially symmetric functions. It is therefore reasonable to believe that beyond the self-similar region $r \gg \sqrt{t}$ the behavior changes into a function of $r/\sqrt{t}$ with fast decay at $+\infty$ that yields finiteness of the second moment. To obtain a first global approximation, we simply cut-off the bubble (2.2) beyond the self-similar zone. We introduce a further parameter $\alpha(t)$ and set

$$u_1(x,t) = \frac{\alpha(t)}{\lambda^2} U_0 \left( \frac{|x - \xi|}{\lambda} \right) \chi(x,t),$$

where we denote

$$\chi(x,t) = \chi_0 \left( \frac{|x - \xi|}{\sqrt{t}} \right)$$

with $\chi_0(s)$ a smooth cut-off function such that

$$\chi_0(s) = \begin{cases} 1 & \text{if } s \leq 1 \\ 0 & \text{if } s \geq 2. \end{cases}$$

The reason why we introduce the parameter $\alpha(t)$ is because the total mass of the actual solution should equal $8\pi$ for all $t$. For the moment let us just impose

$$\int_{\mathbb{R}^2} u_1(x,t) dx = 8\pi.$$

From a direct computation we arrive to the relation $\alpha = \alpha_0$ where

$$\alpha_0(t) = 1 + a \frac{\lambda^2}{t} (1 + o(1)), \quad a = 2 \int_0^\infty \frac{1 - \chi_0(s)}{s^3} ds.$$

Next we will obtain an approximate value of the scaling parameter $\lambda(t)$ that is consistent with the presence of a solution $u(x,t) \approx u_1^0(x,t)$ where $u_1^0$ is the function $u_1$ in (2.5) with $\alpha = \alpha_0$. Let us consider the “error operator”

$$S(u) = -u_t + E(u)$$

where $E(u)$ is defined in (2.3). We have the following well-known identities, valid for an arbitrary function $\omega(x)$ of class $C^2(\mathbb{R}^2)$ with finite mass and $D^2 \omega(x) = O(|x|^{-4-\sigma})$ for large $|x|$. We have

$$\int_{\mathbb{R}^2} |x|^2 \omega(x) dx = 4M - \frac{M^2}{2\pi}, \quad M = \int_{\mathbb{R}^2} \omega(x) dx$$

and

$$\int_{\mathbb{R}^2} x \omega(x) dx = 0, \quad \int_{\mathbb{R}^2} \omega(x) dx = 0.$$
Let us recall the simple proof of (2.7). Integrating by parts on finite balls with large radii and using the behavior of the boundary terms we get the identities

\[
\int_{\mathbb{R}^2} |x|^2 \Delta \omega \, dx = 4M,
\]
\[
\int_{\mathbb{R}^2} |x|^2 \nabla \cdot (\nabla (-\Delta)^{-1}) \omega \, dx = 2 \int_{\mathbb{R}^2} x \cdot \nabla (-\Delta)^{-1} \omega \, dx
\]
\[
= \frac{1}{\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \omega(x) \omega(y) \frac{x \cdot (x - y)}{|x - y|^2} \, dx \, dy
\]
\[
= \frac{1}{2\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \omega(x) \omega(y) \frac{(x - y) \cdot (x - y)}{|x - y|^2} \, dx \, dy
\]
\[
= \frac{M^2}{2\pi}
\]

and then (2.7) follows. The proof of (2.8) is even simpler. For a solution \(u(x,t)\) of (2.1) we then get

\[
\frac{d}{dt} \int_{\mathbb{R}^2} u(x,t) |x|^2 \, dx = 4M - \frac{M^2}{2\pi}, \quad M = \int_{\mathbb{R}^2} u(x,t) \, dx. \tag{2.9}
\]

At this point we also point out that the first moments, namely the center of mass of \(u\) in space is preserved since

\[
\frac{d}{dt} \int_{\mathbb{R}^2} u(x,t) x_i \, dx = 0.
\]

In particular, if \(u(x,t)\) is sufficiently close to \(u_0^1(x,t)\) and since \(\int_{\mathbb{R}^2} u_1(x,t) \, dx = 8\pi\) we get the approximate validity of the identity

\[
\frac{d}{dt} \int_{\mathbb{R}^2} u_1(x,t) |x|^2 \, dx = 0.
\]

This means

\[
I(t) := \int_{\mathbb{R}^2} \frac{\alpha_0}{\lambda^2} U_0 \left( \frac{x - \xi}{\lambda} \right) \chi_0 \left( \frac{|x - \xi|}{\sqrt{t}} \right) |x|^2 \, dx = \text{constant}.
\]

We readily check that for some constant \(\kappa\)

\[
I(t) = 16\pi\lambda^2 \int_0^{\sqrt{t}} \frac{\rho^2 d\rho}{(1 + \rho^2)^2} + \kappa + o(1) = 16\pi\lambda^2 \log \frac{\sqrt{t}}{\lambda} + \kappa + o(1).
\]

Then we conclude that \(\lambda(t)\) approximately satisfies

\[
\lambda^2 \log t = c^2 = \text{constant}
\]

and hence we get at main order

\[
\lambda(t) = \frac{c}{\sqrt{\log t}}.
\]

We also notice that the center of mass is preserved for a true solution, thanks to (2.9):

\[
\frac{d}{dt} \int_{\mathbb{R}^2} xu(x,t) \, dx = 0,
\]

since the center of mass of \(u_1(x,t)\) is exactly \(\xi(t)\) we then get that approximately

\[
\xi(t) = \text{constant} = q
\]
2.2. First error and improvement of approximation. We consider as a first approximation to a solution to (2.1) the function \( u_1(x, t) \) defined by (2.5).

Motivated by the previous considerations we introduce the hypotheses that we make on the parameters \( \lambda(t) > 0, \xi(t) \in \mathbb{R}^2 \) and \( \alpha_1(t) \) in (2.5) that satisfy
\[
\lambda(+\infty) = 0, \quad \alpha_1(+\infty) = 0.
\]

We let
\[
\lambda_*(t) = \frac{1}{\sqrt{\log t}}
\]
For fixed numbers \( M \geq 1, t_0 > 0 \) which we will later take sufficiently large, and a small \( \sigma > 0 \) we assume the following bounds for the derivatives of parameters hold for all \( t \in (t_0, +\infty) \).
\[
|\dot{\lambda}(t)| \leq M|\dot{\lambda}_*(t)|, \quad |\dot{\alpha}_1(t)| \leq M\frac{\lambda^2}{t^2}, \quad |\dot{\xi}(t)| \leq M\frac{1}{t^{1+\sigma}}.
\]

We let
\[
\lambda_1(t) = \lambda_*(t), \quad \alpha_1(t) = \alpha_1_0 + \frac{1}{2} \alpha_1 \frac{|x - \xi|}{t^2}.
\]

We will find an expression for the error of approximation \( S(u_1) \) where \( S(u) \) is the error operator (2.6), and then will build a modification \( u_2(x, t) = u_1(x, t) + \varphi_1(x, t) \) of \( u_1 \) such that the associated error gets reduced beyond the self-similar region.

For the sake of computation, it is convenient to rewrite \( u_1(x, t) \) in (2.5) in the form
\[
u_1(x, t) = \frac{\alpha}{\lambda^2} U_0(y) \chi(y, t), \quad \chi(y, t) = \chi_0 \left( \frac{\lambda|y|}{\sqrt{t}} \right), \quad y = \frac{x - \xi}{\lambda}.
\]

We compute
\[
S(u_1) = -\partial_t u_1 + \mathcal{E}(u_1) = S_1 + S_2 + \mathcal{E}(u_1)
\]
where
\[
S_1 = -\dot{\alpha} \frac{\lambda^2}{2} U_0(y) \chi + \frac{\lambda^2}{\lambda^2} (2U_0(y) + y \cdot \nabla_y U_0(y)) \chi + \frac{1}{2} \alpha \frac{|x - \xi|}{t^2} U_0 \chi_0 \left( \frac{|x - \xi|}{t} \right)
\]
\[
S_2 = \frac{\alpha}{\lambda^3} \dot{\xi} \cdot \nabla_y U_0(y) \chi - \frac{\alpha}{\lambda^3} \dot{\xi} \cdot \frac{\chi_0}{|t|} U_0(y) \chi_0 \left( \frac{|x - \xi|}{t} \right),
\]
\[
y = \frac{x - \xi(t)}{\lambda(t)}
\]
and, we recall, \( \mathcal{E}(u) = \Delta u - \nabla \cdot (u \nabla (-\Delta)^{-1} u) \). Using the notation
\[
V(z) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \log \frac{1}{|z - y|} U_0(z, t) \chi(y, t) \, dy,
\]
we decompose
\[
\mathcal{E}(u_1) = \mathcal{E}^o + \mathcal{E}^i
\]
where
\[
\mathcal{E}^i = \lambda^{-4} \left[ \alpha(\alpha - 1) U_0^2 \chi - \alpha(\alpha - 1) \nabla_y U_0 \nabla_y V_0 \chi \right]
\]
\[
\mathcal{E}^o = \lambda^{-4} \left[ 2\alpha \nabla_y U_0 \cdot \nabla_y \chi + \alpha U_0 \Delta_y \chi + U_0^2 \alpha^2 \chi(\chi - 1) \right.
\]
\[
- \alpha^2 U_0 \nabla_y \chi \nabla_y V + \alpha \nabla_y U_0 \nabla_y (V - V_0) \chi
\]
\[
- \alpha(\alpha - 1) \nabla_y U_0 \nabla_y (V - V_0) \chi
\]
The superscripts \( i \) and \( o \) respectively refer to “inner” and “outer” parts of the error that will later be dealt with separately. We also decompose

\[
S_1 = S_i^1 + S_o^1
\]

where

\[
\begin{align*}
S_i^1 &= -\frac{\dot{\alpha}}{\lambda^2} U_0 \chi + \alpha \frac{\lambda}{\lambda^7} (2U_0 + y \cdot \nabla_y U_0) \chi \\
S_o^1 &= \frac{1}{2} \alpha \frac{|x - \xi|}{\lambda^2 t^{\frac{3}{2}}} U_0 \chi(\frac{|x - \xi|}{\sqrt{t}}).
\end{align*}
\]

(2.13)

Next we introduce a correction \( \varphi_o^1(x,t) \) as in (2.11) that eliminates the largest terms of the error \( E^o + S_{i1}^o \). We see that, at main order,

\[
E^o + S_{i1}^o \approx \lambda^2 t^{\frac{3}{2}} h(\zeta), \quad \zeta = \frac{|x - \xi|}{\sqrt{t}},
\]

where

\[
h(\zeta) = \frac{8}{\zeta^4} \left[ \chi'' - \frac{3}{\zeta} \chi''(\zeta) + \frac{\zeta}{2} \chi'(\zeta) \right].
\]

In agreement with the approximate expression (2.4) for the remote regime of (2.1), we look for the correction \( \varphi_o^1 \) in the form

\[
\varphi_1(x,t) = \lambda^2 \tilde{\varphi}_1(\frac{|x - \xi|}{\sqrt{t}}),
\]

(2.14)

where \( \tilde{\varphi}_1(r,t) \) solves the radial heat equation in dimension 6:

\[
\begin{cases}
\partial_t \tilde{\varphi}_1 = \partial_r^2 \tilde{\varphi}_1 + \frac{5}{r} \partial_r \tilde{\varphi}_1 + \frac{1}{t^{\frac{3}{2}}} h \left( \frac{r}{\sqrt{t}} \right), \\
\tilde{\varphi}_1(r,0) = 0.
\end{cases}
\]

(2.15)

The solution \( \tilde{\varphi}_1(r,t) \) to problem (2.15) can be expressed in self-similar form as

\[
\tilde{\varphi}_1(r,t) = \frac{1}{t^{\frac{3}{2}}} g(\zeta), \quad \zeta = \frac{r}{\sqrt{t}}.
\]

We find for \( g \) the equation

\[
g'' + \frac{5}{\zeta} g' + \frac{\zeta}{2} g' + 2g + h(\zeta) = 0, \quad \zeta \in (0, \infty).
\]

(2.16)

Using that the function \( \frac{1}{\zeta^4} \) is in the kernel of the homogenous equation, we find the explicit solution of (2.16),

\[
g_0(\zeta) = -\frac{1}{\zeta^4} \int_0^\zeta x^3 e^{-\frac{1}{4}x^2} \int_0^x h(y)e^{\frac{1}{2}y^2} y dy dx.
\]

To find the solution \( \varphi_1 \) with initial condition zero we define

\[
g(\zeta) = g_0(\zeta) + \frac{1}{8} \tilde{z}(\zeta) I,
\]

(2.17)

where

\[
\tilde{z}(\zeta) = \frac{1}{\zeta^4} \int_0^{\zeta} x^3 e^{-\frac{1}{4}x^2} dx
\]

is a second solution of the homogeneous equation, linearly independent of \( \frac{1}{\zeta^4} \) and

\[
I = \int_0^{\infty} x^3 e^{-\frac{1}{4}x^2} \int_0^x h(y)e^{\frac{1}{2}y^2} y dy dx.
\]
We observe that
\[ g(\xi) = O(e^{-\frac{1}{2}\xi^2}) \quad \text{as} \quad \xi \to +\infty, \quad \text{(2.18)} \]
which makes the solution (2.17) the only one with decay faster than \( O(\xi^{-4}) \) as \( \xi \to +\infty \). An explicit calculation gives that \( I = -8 \), and therefore
\[ \varphi_1(\xi(t), t) = -\frac{\lambda(t)^2}{4t^2}, \quad \text{(2.19)} \]
an identity that will play a crucial role in later computations.

We take then as the basic approximation the function \( u_2(x, t) \) in the form (2.11) defined as
\[ u_2(x, t) = \frac{\alpha(t)}{\lambda^2} U_0 \left( \frac{x - \xi(t)}{\lambda(t)} \right) \chi_0 \left( \frac{|x - \xi(t)|}{\sqrt{t}} \right) + \varphi_1(x, t), \quad \text{(2.20)} \]
where \( \varphi_1 \) is defined by (2.14). Accordingly, we write
\[ \psi_1 = (-\Delta)^{-1} \varphi_1. \]
The direct computation of the new error, using estimate (2.18) that inherits Gaussian decay for \( \varphi_1(x, t) \), yields the validity of the following bound.

**Lemma 2.1.** Let \( u_2 \) be given by (2.20) with \( \varphi_1(x, t) \) defined as in (2.14). The error of approximation \( S(u_2) \) can be expressed as
\[ S(u_2) = E^i + S_1^i + S_2 + R \]
where \( S_1^i, S_2 \) are defined in (2.12), (2.13) and
\[ |R(x, t)| \leq C \frac{M^2}{t^3(\log t)^2} e^{-\frac{c|x-y|^2}{t}}, \]
for universal constants \( c, C > 0 \). \( M \) is the number in constraints (2.10).

**The inner-outer gluing system.** Next we set up an ansatz for a solution \( u(x, t) \) of Equation (2.1) which goes along the decomposition we have just made for
\[ u_2(x, t) = \frac{1}{\lambda^2} U_0(y) \chi(x, t) + \varphi_1(x, t) \]
in which we made a distinction between inner and outer parts of the remainder. We look for a solution \( u(x, t) \) of (2.1) of the form
\[ u(x, t) = u_2(x, t) + \frac{1}{\lambda^2} \phi^i(y, t) \chi(x, t) + \phi^o(x, t), \quad y = \frac{x - \xi}{\lambda} \quad \text{(2.21)} \]
where \( \chi(x, t) = \chi_0 \left( \frac{|x - \xi|}{\sqrt{t}} \right) \) and the superscripts \( i \) and \( o \) designate inner and outer parts of the remainder. We compute
\[ -u_t = -\partial_t u_2 + \frac{\dot{\lambda}}{\lambda^3} (2\phi^i + \nabla_y \phi^i \cdot y) \chi - \nabla_y \phi^i \cdot \frac{\xi}{\lambda^3} \chi - \frac{1}{\lambda^2} \phi^i_t \chi - \frac{1}{\lambda^2} \phi^i \chi_t - \phi^o \]
We let
\[ \psi^i(y, t) = (-\Delta_y)^{-1} \phi^i = \frac{1}{2\pi} \int_{\mathbb{R}^2} \log \frac{1}{|y - z|} \phi^i(z, t) \, dz, \quad y = \frac{x - \xi}{\lambda} \]
\[ v_2(x, t) = (-\Delta_x)^{-1} u_2 = \frac{1}{2\pi} \int_{\mathbb{R}^2} \log \frac{1}{|x - w|} u_2(w, t) \, dw \]
and then we take
\[ v(x, t) = v_2(x, t) + \psi^1(y, t)\chi + \psi^o(x, t) \]

We then get the following system of equations for $\phi^i, \psi^i, \phi^o, \psi^o$:
\[
\begin{align*}
\lambda^2 \partial_t \phi^o &= \Delta_y \phi^i - \nabla_y \phi^i \nabla_y \phi^o - \nabla_y \psi^i \nabla_y \phi^1 + 2\lambda^2 \nabla_y \phi^o + 2\lambda^2 \nabla_y \phi^1 - \nabla y[U_0\chi] \nabla x \psi^o - \nabla y[U_0\chi] \nabla x \psi^1 + E_{in} \\
-\Delta_y \psi^i &= \phi^i 
\end{align*}
\]

where
\[
L_{out} = \Delta_y \phi^o - \nabla_y \phi^o \nabla y \phi^o \left( \frac{x - \xi(t)}{\lambda(t)} \right) 
\]

\[
G(\phi^i, \phi^o, \psi^i, \psi^o, \lambda, \alpha) = A_1[\phi^o] + A_2[\psi^o] + B_1[\phi^i] + B_2[\psi^i] + E_{out} + N(\phi^i, \phi^o, \psi^i, \psi^o) 
\]

with
\[
A_1[\phi^o] = 2u_2 \phi^o(1 - \eta) - \nabla_x \phi^o \nabla_x \bar{v}_2 + 2[\alpha - 1] \lambda^{-2} U_0 \eta + \phi_1 \phi^o \\
A_2[\psi^o] = -\nabla_x u_2 \nabla_x \psi^o(1 - \eta) - \lambda^{-2} \nabla_x \psi^o \nabla y [(\alpha - 1) U_0 \chi + \lambda^2 \phi_1] \\
B_1[\phi^i] = \frac{\lambda}{\lambda^3} - \frac{\xi}{\lambda^3} \nabla_y \phi^i \eta + (2\phi^i + \nabla_y \phi^i y) \eta + \lambda^{-2} (\Delta_y \eta - \eta_1) \phi^i \\
&+ 2\lambda^{-2} \nabla_x \phi^i \nabla_x \eta - \lambda^{-2} \phi^i \nabla_x \eta \nabla_x v_2 + 2\lambda^{-4} U_0(1 - \eta) + 2\alpha - 1) U_0 \eta] \phi^i \\
B_2[\psi^i] = -\psi^i \nabla_x u_2 \nabla_x \eta - \lambda^{-4} \nabla_y \psi^i \nabla y |U_0(1 - \chi) + (\alpha - 1) U_0 \chi + \lambda^2 \phi_1| \eta 
\]

In the above expressions $\bar{v}_2$ is given by
\[ -\Delta_y \bar{v}_2 = \lambda^{-2} U_0(y)(\chi - 1) + (\alpha - 1) \lambda^{-2} U_0(y)\chi + \phi_1, \]

and $E_{in}$ is given by
\[ E_{in} = \alpha(\alpha - 1) U_0^2 - \alpha(\alpha - 1) \nabla U_0 \nabla V_0 - \lambda^2 \partial_t U_0 \chi + \alpha \lambda \lambda(2 U_0 + y \cdot \nabla U_0) \chi \\
+ \alpha \lambda \xi \cdot \nabla U_0 \chi. \]

It follows that $E_{out}$ satisfies the estimate
\[ |E_{out}| \leq C \begin{cases} \frac{1}{t^{3 \log^2 t}} & r \leq \sqrt{t} \\
\frac{1}{t \log^{\frac{1}{2}} t^2} & r \geq \sqrt{t}. \end{cases} \]

Finally, $N(\phi^i, \phi^o, \psi^i, \psi^o)$ are the nonlinear terms:
\[ N(\phi^i, \phi^o, \psi^i, \psi^o) = -\nabla \Phi \nabla \Psi + 2 \Phi^2 \]

with
\[ \Phi = \lambda^{-2} \phi^i \left( \frac{x - \xi}{\lambda} \right) \chi(x, t) + \phi^o(x, t) \\
\Psi = \psi^i \chi + \psi^o. \]
Instead of solving the system (2.22)-(2.23) directly, we consider the following modification of (2.22)

\[
\begin{aligned}
\lambda^2 \partial_t \phi^i &= \Delta_y \phi^i - \nabla_y \phi^i \nabla_y V_0 - \nabla_y \phi^i \nabla_y U_0 + 2U_0 \phi^i \\
&\quad + H(\phi^o, \psi^o, \lambda, \alpha_1, \xi) - \sum_{j=1}^4 c_j(t) \tilde{Z}_j \\
-\Delta_y \psi^i &= \phi^i
\end{aligned}
\]  

where

\[
H(\phi^o, \psi^o, \lambda, \alpha_1, \xi) = 2\lambda^2 U_0 \chi \phi^o + 2\lambda^2 U_0 \chi \phi_1 \\
- \lambda \nabla_y [U_0 \chi] \nabla_x \psi^o - \lambda \nabla_y [U_0 \chi] \nabla_x \psi_1, \\
+ \alpha(\alpha - 1) U_0^2 - \alpha(\alpha - 1) \nabla U_0 \nabla V_0
\]  

(2.25)

\[
\tilde{Z}_j(y) = Z_j(y) \chi(y)
\]

and

\[
\begin{aligned}
Z_0(y) &= 2U_0(y) + \nabla U_0(y) y, \\
Z_1(y) &= \partial_y U_0(y), \\
Z_2(y) &= \partial_y U_0(y), \\
Z_3(y) &= U_0(y).
\end{aligned}
\]  

(2.26)

We consider then together (2.23) and (2.24) for \( t > t_0 \), where \( t_0 \) is a large fixed number, and we impose initial conditions at \( t_0 \) of the form \( \phi^i(\cdot, t_0) = 0, \psi^i(\cdot, t_0) = \phi^o_0 \), where \( \phi^o_0 \) is a function whose properties will be given later on.

In (2.24) we define \( c_j(t) \) by the requirement that

\[
h = H(\phi^o, \psi^o, \lambda, \alpha_1, \xi) - \sum_{j=1}^4 c_j(t) \tilde{Z}_j
\]

satisfies the four conditions

\[
\begin{aligned}
\int_{\mathbb{R}^2} h(y, t) \, dy &= 0, \\
\int_{\mathbb{R}^2} h(y, t)|y|^2 \, dy &= 0, \\
\int_{\mathbb{R}^2} h(y, t)y_j \, dy &= 0, \quad j = 1, 2,
\end{aligned}
\]

for all \( t > t_0 \).

To find then a solution of the original problem (2.1) it is sufficient to find a solution \( \phi^i, \psi^i, \phi^o, \psi^o \) of the system system comprised by (2.23)-(2.24), and then to find \( \lambda, \alpha = 1 + \alpha_1 \) and \( \xi \) so that

\[
\begin{aligned}
c_0 &= \alpha \lambda \dot{\lambda}, & c_1 &= \alpha \lambda \dot{\xi}_1, \\
c_3 &= -\lambda^2 \dot{\alpha}, & c_2 &= \alpha \lambda \dot{\xi}_2.
\end{aligned}
\]

for all \( t > t_0 \).
For the proof of the main theorem, we will use the following norms. For the inner problem we define

\[ \| \phi^i \|_{\lambda, \alpha, \mu, \sigma} = \sup_{|y| \leq \sqrt{t}} \{ t^\alpha (1 + |y|)^{2+\sigma} |\phi^i(y, t)| + (1 + |y|)^{4+\sigma} |\nabla \phi^i(y, t)| \} \]

\[ + \sup_{|y| \geq \sqrt{t}} \{ t^{\sigma-1} (1 + |y|)^{5+\sigma} |\phi^i(y, t)| + (1 + |y|)^{6+\sigma} |\nabla \phi^i(y, t)| \} \]

\[ \| h^i \|_{\lambda, \alpha, \mu, \sigma} = \sup \{ t^\sigma (1 + |y|)^{5+\sigma} |h^i(y, t)| \} \]

where \( \nu > 0, \mu \geq 0, \) and \( \sigma > 0. \)

For the outer problem we use the norms

\[ \| \phi^o \|_{\lambda, \alpha, a, b, \sigma} = \sup \left\{ t^\alpha (1 + |y|)^{1+\sigma} \left( \frac{|x - \xi|}{\lambda} + 1 \right)^{b-2} + \lambda^2 t^{a-1} (1 + |y|)^{1+\sigma} \left( \frac{|x - \xi|}{\lambda} + 1 \right)^{b+1} \right\} |\phi^o(x, t)| \]

\[ + \sup \left\{ t^\alpha (1 + |y|)^{1+\sigma} \left( \frac{|x - \xi|}{\lambda} + 1 \right)^{b-1} + \lambda^2 t^{a-1} (1 + |y|)^{1+\sigma} \left( \frac{|x - \xi|}{\lambda} + 1 \right)^{b+1} \right\} |\nabla \phi^o(x, t)| \]

\[ \| h \|_{\lambda, \alpha, a, b, \sigma} = \sup \{ \lambda^2 t^{a} (1 + |y|)^{1+\sigma} \left( \frac{|x - \xi|}{\lambda} + 1 \right)^{b} \} |h(x, t)|, \]

where \( 2 < b < 6, 1 < a < 3, \mu \geq 0. \)

Proof of Theorem 1. We let \( \sigma > 0 \) be a small number. We work in both equations (2.24) and (2.23) with \( t \geq t_0 \) and \( t_0 \) a large constant. In (2.24) we impose zero initial condition and in (2.23) we put initial condition \( \phi^o(x, t_0) = \phi_0^o(x) \) a function with small norm (1.3). We assume that \( \lambda, \alpha_2 \) and \( \xi \) satisfy the conditions (2.10). Given a function \( \phi^o \) with \( \| \phi^i \|_{\lambda, \alpha, \mu, \sigma} \) small we solve (2.23) using Theorem 3 and the Banach fixed point theorem, and we find a solution \( \phi^o \) with the estimate

\[ \| \phi^o \|_{\infty, 2, 2, 0} \lesssim \| \phi^i \|_{\lambda, 1, 2, \sigma} + \| \phi_0^o \|_{\lambda, \sigma}. \]

Here we have used in a crucial manner the fact that

\[ \phi_0^o(x) = O(|x|^{-4-\sigma}) \quad \text{as} \quad |x| \to \infty, \]

since from here and Duhamel’s formula we find that \( \phi^o(x, t) = O(t^{-2-\frac{\sigma}{2}}). \)

It is straightforward to check that \( \phi^o, \psi^o \) are Lipschitz functions of \( \phi^i \). Inserting these functions in (2.24) we solve now for \( \phi^i \), having \( \lambda, \alpha_2 \) and \( \xi \) as given parameters. The functions \( c_1(t) \) in (2.24) are chosen such that \( H(\phi^o, \psi^o, \phi_1, \psi_1) \) satisfies the four conditions in Theorem 2. Using that theorem and the Banach fixed point theorem, we find that (2.24) has a solution \( \phi^i \) with \( \| \phi^i \|_{\lambda, 1, 2, \sigma} \) small. To find a solution to the problem we need the following conditions to be satisfied

\[ \lambda^2 \hat{\alpha} \int_{\mathbb{R}^2} U_0 x \, dy = \int_{\mathbb{R}^2} H(\phi^o, \psi^o, \lambda, \alpha_1, \xi) \, dy \]  

\[ \alpha \lambda \hat{\lambda} = \frac{1}{\int_{\mathbb{R}^2} |y| \, dy} \int_{\mathbb{R}^2} H(\phi^o, \psi^o, \lambda, \alpha_1, \xi) |y|^2 \, dy \]

\[ \alpha \lambda \hat{\xi} = \frac{1}{\int_{\mathbb{R}^2} Z_1 |y| \, dy} \int_{\mathbb{R}^2} H(\phi^o, \psi^o, \lambda, \alpha_1, \xi) y \, dy, \]

where \( H \) is given by (2.25). At main order, after some computation we find the equation

\[ -\lambda^2 \hat{\alpha}_1 8\pi + 2\lambda^2 \int_{\mathbb{R}^2} U_0 \phi - \lambda^2 \int_{\mathbb{R}^2} \nabla U_0 \cdot \nabla \psi_1 = \frac{1}{t^2 \log^2 t} N_1(\lambda, \hat{\alpha}_1, \hat{\xi}). \]
where \( \mathcal{N}_1 \) is a uniformly bounded operator, Lipschitz in its arguments for their associated topologies in (2.10). After an integration by parts, and integrating in time from \( t \) to \( +\infty \), we find that at main order

\[
\alpha_1(t) - \frac{\lambda^2}{4t} + \frac{1}{t \log^2 t} \mathcal{N}_2(\lambda, \dot{\alpha}_1, \dot{\xi}) = 0.
\]

Replacing this in (2.28) gives at main order

\[
\lambda \dot{\lambda} = -\frac{1}{2} \lambda^2 t (1 + O(\frac{1}{\log t})),
\]

and solving for \( \lambda \) gives the desired rate after an application of contraction mapping principle to system (2.27), (2.28), (2.29). The equations for \( \lambda \) and \( \xi \) is have to be solved by fixing their initial conditions independently of \( \phi_0 \). The fact that perturbative initial condition \( \phi_0 \) was arbitrary gives the stability of the blow-up, since if we assume \( \int_{\mathbb{R}^2} \phi_0 = 0 \) then necessarily \( \alpha_1(t_0) = 0 \) which precisely amounts to the mass of the full initial condition to be exactly \( 8\pi \). We check that for the topology we are using in the initial conditions we can represent an arbitrary perturbation in this form. This concludes the proof.

\[ \square \]

3. Preliminaries for the linear theory

3.1. The Liouville equation. Let

\[
U_0(y) = \frac{8}{(1 + |y|^2)^2} \quad (3.1)
\]

\[
V_0(y) = \log \frac{8}{(1 + |y|^2)^2}
\]

with \( y \in \mathbb{R}^2 \) and note that

\[
\Delta V_0 + e^{V_0} = 0 \quad \text{in} \quad \mathbb{R}^2.
\]

The linearization around of \( V_0 \) is given by the operator

\[
\psi \mapsto \Delta \psi + U_0 \psi
\]

and we note that the following explicit functions

\[
\begin{cases}
  z_0(y) = \nabla V_0(y) \cdot y + 2 \\
  z_j(y) = \partial_{y_j} V_0(y), \quad j = 1, 2
\end{cases} \quad (3.2)
\]

are elements in the kernel of \( \Delta + U_0 \). These are the only bounded elements in the kernel.

3.2. Stereographic projection. Let \( \Pi : S^2 \setminus \{(0,0,1)\} \to \mathbb{R}^2 \) denote the stereographic projection

\[
\Pi(y_1, y_2, y_3) = \left( \frac{y_1}{1-y_3}, \frac{y_2}{1-y_3} \right).
\]

For \( \varphi : \mathbb{R}^2 \to \mathbb{R} \) we write

\[
\tilde{\varphi} = \varphi \circ \Pi, \quad \varphi : S^2 \setminus \{(0,0,1)\} \to \mathbb{R}.
\]
Let $U_0$ be given by (3.1). Then we have the following formulas

\[
\int_{S^2} \tilde{\varphi} = \frac{1}{2} \int_{\mathbb{R}^2} \varphi U_0 \\
\int_{S^2} \tilde{U}_0 |\nabla_{S^2} \tilde{\varphi}|^2 = \int_{\mathbb{R}^2} U_0 |\nabla_{\mathbb{R}^2} \varphi|^2 \\
\frac{1}{2} \tilde{U}_0 \Delta_{S^2} \tilde{\varphi} = (\Delta_{\mathbb{R}^2} \varphi) \circ \Pi.
\]

The linearized Liouville equation for $\phi, f : \mathbb{R}^2 \to \mathbb{R}$

\[
\Delta \phi + U_0 \phi + U_0 f = 0 \quad \text{in} \ \mathbb{R}^2
\]

is transformed into

\[
\Delta_{S^2} \tilde{\phi} + 2 \tilde{\phi} + 2 \tilde{f} = 0 \quad \text{in} \ S^2 \setminus \{(0,0,1)\}.
\]

### 3.3. A quadratic form.

**Lemma 3.1.** Let $\phi : \mathbb{R}^2 \to \mathbb{R}$ satisfy

\[
|\phi(y)| \leq \frac{1}{(1 + |y|)^{2+\sigma}},
\]

with $0 < \sigma < 1$, and

\[
\int_{\mathbb{R}^2} \phi \, dy = 0.
\]

There are constants $c_1 > 0, c_2 > 0$ such that

\[
c_1 \int_{\mathbb{R}^2} U_0 g^2 \leq \int_{\mathbb{R}^2} \phi g \leq c_2 \int_{\mathbb{R}^2} U_0 g^2
\]

where

\[
g = \frac{\phi}{U_0} - (-\Delta^{-1})\phi + c
\]

and $c \in \mathbb{R}$ is chosen so that

\[
\int_{\mathbb{R}^2} g U_0 = 0.
\]

**Proof.** We set

\[
\psi_0 = (-\Delta)^{-1} \phi
\]

and then note that since $\int_{\mathbb{R}^2} \phi = 0$ we have

\[
|\psi_0(y)| + (1 + |y|)|\nabla \psi_0(y)| \lesssim \frac{1}{(1 + |y|)^{\sigma}}.
\]

From $g = \frac{\phi}{U_0} - \psi_0 + c$ we find the estimate

\[
|g(y)| \lesssim (1 + |y|)^{2-\sigma}.
\]

Let $\psi = \psi_0 - c$ and note that

\[
-\Delta \psi - U_0 \psi = U_0 g \quad \text{in} \ \mathbb{R}^2.
\]

We transform $\tilde{g} = g \circ \Pi, \tilde{\psi} = \psi \circ \Pi$ and write this equation in $S^2$ as

\[
-\Delta_{S^2} \tilde{\psi} - 2 \tilde{\psi} = 2 \tilde{g} \quad \text{in} \ S^2.
\]
Since $\phi = U_0(g + \psi)$ we get

$$
\int_{\mathbb{R}^2} \phi g = \int_{\mathbb{R}^2} U_0(g + \psi) g = \frac{1}{2} \int_{S^2} \tilde{g}^2 + \psi \tilde{g},
$$

Multiplying (3.3) by $\tilde{\psi}$ we find that

$$
\int_{S^2} \tilde{g} \tilde{\psi} = \frac{1}{2} \int_{S^2} |\nabla \tilde{s} \tilde{\psi}|^2 - \int_{S^2} \tilde{\psi}^2
$$

and hence

$$
\int_{\mathbb{R}^2} \phi g = \frac{1}{2} \int_{S^2} \tilde{g}^2 + \frac{1}{4} \int_{S^2} |\nabla \tilde{s} \tilde{\psi}|^2 - \frac{1}{2} \int_{S^2} \tilde{\psi}^2.
$$

We recall that the eigenvalues of $-\Delta$ on $S^2$ are given by $\{k(k + 1) \mid k \geq 0\}$. The eigenvalue 0 has a constant eigenfunction and the eigenvalue 2 has eigenspace spanned by the coordinate functions $\pi_i(x_1, x_2, x_3) = x_i$, for $(x_1, x_2, x_3) \in S^2$ and $i = 1, 2, 3$. Let $(\lambda_j)_{j \geq 0}$ denote all eigenvalues, repeated according to multiplicity, with $\lambda_0 = 0$, $\lambda_1 = \lambda_2 = \lambda_3 = 2$, and let $(e_j)_{j \geq 0}$ denote the corresponding eigenfunctions so that they form an orthonormal system in $L^2(S^2)$, and $e_1, e_2, e_3$ are multiples of the coordinate functions $\pi_1, \pi_2, \pi_3$. We decompose $\tilde{\psi}$ and $\tilde{g}$:

$$
\tilde{\psi} = \sum_{j=0}^{\infty} \tilde{\psi}_j e_j, \quad \tilde{g} = \sum_{j=0}^{\infty} \tilde{g}_j e_j,
$$

where

$$
\tilde{\psi}_j = \langle \tilde{\psi}, e_j \rangle_{L^2(S^2)}, \quad \tilde{g}_j = \langle \tilde{g}, e_j \rangle_{L^2(S^2)}.
$$

Then

$$
\int_{\mathbb{R}^2} \phi g = \frac{1}{2} \int_{S^2} \tilde{g}^2 + \frac{1}{4} \sum_{j=0}^{\infty} (\lambda_j - 2) \tilde{\psi}_j^2
$$

$$
= \frac{1}{2} \int_{S^2} \tilde{g}^2 - \frac{1}{2} \psi_0^2 + \frac{1}{4} \sum_{j=4}^{\infty} (\lambda_j - 2) \tilde{\psi}_j^2.
$$

Equation (3.3) gives us that

$$
\tilde{\psi}_j = \frac{2}{\lambda_j - 2} \tilde{g}_j, \quad j \notin \{1, 2, 3\},
$$

and therefore

$$
\int_{\mathbb{R}^2} \phi g = \frac{1}{2} \int_{S^2} \tilde{g}^2 - \frac{1}{2} \psi_0^2 + \sum_{j=4}^{\infty} \frac{1}{\lambda_j - 2} \tilde{g}_j^2
$$

$$
= \frac{1}{2} \sum_{j=1}^{\infty} \tilde{g}_j^2 + \sum_{j=4}^{\infty} \frac{1}{\lambda_j - 2} \tilde{g}_j^2.
$$

But $\int_{\mathbb{R}^2} gU_0 = 0$ which means that $\tilde{g}_0 = 0$ and hence we obtain the conclusion. \(\square\)

**Lemma 3.2.** Suppose that $\phi = \phi(y, t), \ y \in \mathbb{R}^2, \ t > 0$ is a function satisfying

$$
|\phi(y, t)| \leq \frac{1}{(1 + |y|)^{2+\sigma}},
$$
with $0 < \sigma < 1$,

$$\int_{\mathbb{R}^2} \phi(y, t) \, dy = 0, \quad \forall t > 0,$$

and that $\phi$ is differentiable with respect to $t$ and $\phi_t$ satisfies also

$$|\phi_t(y, t)| \leq \frac{1}{(1 + |y|)^{2+\sigma}}.$$

Then

$$\int_{\mathbb{R}^2} \phi_t g = \frac{1}{2} \theta_t \int_{\mathbb{R}^2} \phi g$$

where for each $t$, $g(y, t)$ is defined as

$$g = \frac{\phi}{U_0} - (-\Delta^{-1})\phi + c(t)$$

and $c(t) \in \mathbb{R}$ is chosen so that

$$\int_{\mathbb{R}^2} g(y, t)U_0(y) \, dy = 0.$$

Proof. Using the notation of the previous lemma, we have

$$\int_{\mathbb{R}^2} \phi_t g = \int_{\mathbb{R}^2} U_t(g_t + \psi_t)g = \frac{1}{2} \int_{S^2} (\tilde{g}_t\tilde{g} + \tilde{\psi}_t\tilde{g}).$$

We have

$$-\Delta_{S^2}\tilde{\psi} - 2\tilde{\psi} = 2\tilde{g}, \quad \text{in } S^2.$$

And differentiating in $t$ we get

$$-\Delta_{S^2}\tilde{\psi}_t - 2\tilde{\psi}_t = 2\tilde{g}_t, \quad \text{in } S^2. \quad (3.5)$$

Multiplying by $\tilde{g}$ and integrating we find that

$$\int_{S^2} \tilde{\psi}_t \tilde{g} = -\frac{1}{2} \int_{S^2} \Delta \tilde{\psi}_t \tilde{g} - \int_{S^2} \tilde{g}_t \tilde{g}.$$

Thus

$$\int_{\mathbb{R}^2} \phi_t g = -\frac{1}{4} \int_{S^2} \Delta \tilde{\psi}_t \tilde{g}.$$

Decompose as in (3.4) and find that

$$\int_{\mathbb{R}^2} \phi_t g = \frac{1}{4} \sum_{j=0}^{\infty} \lambda_j (\tilde{\psi}_j)_t \tilde{g}_j.$$

But from (3.5)

$$(\lambda_j - 2)(\tilde{\psi}_j)_t = 2(\tilde{g}_j)_t.$$

If

$$\tilde{g}_j = 0, \quad j = 0, 1, 2, 3,$$

then

$$\int_{\mathbb{R}^2} \phi_t g = \frac{1}{2} \sum_{j=4}^{\infty} \frac{\lambda_j}{\lambda_j - 2} (\tilde{g}_j)_t \tilde{g}_j.$$

\[\square\]
3.4. A Hardy inequality.

**Lemma 3.1.** There exists $C > 0$ such that, for any $R > 0$ large and any $\int_{B_R} gU_0 = 0$

$$\frac{C}{R^2} \int_{B_R} g^2 U_0 \leq \int_{B_R} |\nabla g|^2 U_0.$$

**Proof.** After a stereographic projection and letting $\varepsilon = \frac{1}{R}$, $A_\varepsilon = B_1(0) \setminus B_\varepsilon(0) \subset \mathbb{R}^2$, we need to prove that for $g \in C^1(A_\varepsilon)$ with

$$\int_{A_\varepsilon} g \, dy = 0,$$

we have

$$\int_{A_\varepsilon} g^2 \, dy \leq \frac{C}{\varepsilon^2} \int_{A_\varepsilon} |\nabla g|^2 |y|^4 \, dy.$$

By using polar coordinates it is sufficient to show this for radial functions, which amounts to the statement: for $g \in C^1([\varepsilon, 1])$, if

$$\int_\varepsilon^1 g^2 x \, dx = 0 \tag{3.6}$$

then

$$\int_\varepsilon^1 g^2 x \, dx \leq \frac{C}{\varepsilon^2} \int_\varepsilon^1 g'(x)^2 x^5 \, dx.$$

We write

$$\int_\varepsilon^1 g^2 x \, dx = \frac{1}{2} \int_\varepsilon^1 \frac{d}{dx} (x^2) \, dx \quad g'^2 - \frac{g^2}{2} \varepsilon^2 - \frac{1}{2} \int_\varepsilon^1 gg' x^2 \, dx.$$

One has

$$\int_\varepsilon^1 gg' x^2 \, dx \leq \left( \int_\varepsilon^1 g'^2 x^5 \, dx \right)^{\frac{1}{2}} \left( \int_\varepsilon^1 g^2 x^{-1} \, dx \right)^{\frac{1}{2}}$$

$$\leq \left( \varepsilon^{-2} \int_\varepsilon^1 g'^2 x^5 \, dx \right)^{\frac{1}{2}} \left( \int_\varepsilon^1 g^2 x \, dx \right)^{\frac{1}{2}}$$

$$\leq C \varepsilon^{-2} \int_\varepsilon^1 g'^2 x^5 \, dx + \frac{1}{2} \int_\varepsilon^1 g^2 x \, dx,$$

for some constant $C$. Inserting this inequality in the previous computation gives

$$\int_\varepsilon^1 g^2 x \, dx \leq g(1)^2 - g(\varepsilon)^2 \varepsilon^2 + C \varepsilon^{-2} \int_\varepsilon^1 g'^2 x^5 \, dx. \tag{3.7}$$

We now use (3.6) in the form

$$0 = \int_\varepsilon^1 g(x) x \, dx = \frac{g(1)}{2} \varepsilon^2 - \int_\varepsilon^1 g' x^2 \, dx,$$

and so

$$g(1)^2 \leq 2g(\varepsilon)^2 \varepsilon^4 + 2 \left( \int_\varepsilon^1 g' x^2 \, dx \right)^2.$$
But
\[ \int_{\epsilon}^{1} g'(x)^2 \, dx \leq \left( \int_{\epsilon}^{1} g'^2 x^5 \, dx \right)^{\frac{1}{2}} \left( \int_{\epsilon}^{1} x^{-1} \, dx \right)^{\frac{1}{2}} \leq \left| \log \epsilon \right| \int_{\epsilon}^{1} g'^2 x^5 \, dx \right)^{\frac{1}{2}}. \]
We thus get that
\[ g(1)^2 \leq 2g(\epsilon)^2 \epsilon^4 + 2| \log \epsilon | \int_{\epsilon}^{1} g'^2 x^5 \, dx \]and this combined with (3.7) gives
\[ \int_{\epsilon}^{1} g^2 x \, dx \leq g(\epsilon)^2 (2\epsilon^4 - \epsilon^2) + (C\epsilon^{-2} + 2| \log \epsilon |) \int_{\epsilon}^{1} g'^2 x^5 \, dx. \]
For \( \epsilon > 0 \) small this gives the desired estimate.

\[ \square \]

4. Inner Problem

The relevant linear equation for \( \phi^i \) appearing in (2.24) is
\[ \lambda^2 \partial_t \phi^i = \Delta y \phi^i - \nabla_y \nabla y V_0 - \nabla_y \psi^i \nabla y U_0 + 2U_0 \phi^i + h, \]
where we recall that \( \psi^i = (-\Delta)^{-1} \phi^i \).

We change the time variable
\[ \tau = \int_{\tau_0}^{t} \frac{1}{\lambda^2(s)} \, ds \]
and note that \( \tau \sim t \log t \). Then this equation can be written as
\[ \partial_{\tau} \phi = \nabla \cdot \left[ U \nabla \left( \frac{\phi}{U_0} - (-\Delta)^{-1} \phi \right) \right] + h, \quad \text{in} \ \mathbb{R}^2 \times (\tau_0, \infty), \]
where
\[ (-\Delta)^{-1} \phi(x, t) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \log \left( \frac{1}{|x-y|} \right) \phi(y, t) \, dy, \]
and \( \tau_0 \) is fixed large. We consider it with initial condition
\[ \phi(y, \tau_0) \equiv 0, \quad \text{in} \ \mathbb{R}^2. \]

We assume that
\[ |h(y, \tau)| \leq \frac{1}{\tau^\nu \log^\mu \tau (1 + |y|)^{5+\sigma}} \]
with
\[ 0 < \nu < 3, \quad \mu \in \mathbb{R}, \quad 0 < \sigma < 1. \]

We define
\[ \|h\|_{i, \nu, \mu, \sigma} = \sup \{ \tau^\nu \log^\mu \tau (1 + |y|)^{5+\sigma} |h(y, \tau)| \}. \]
We also assume that
\[ \int_{\mathbb{R}^2} h(y, \tau) \, dy = 0, \]  
(4.1)
\[ \int_{\mathbb{R}^2} h(y, \tau) |y|^2 \, dy = 0, \]  
(4.2)
\[ \int_{\mathbb{R}^2} h(y, \tau) y_j \, dy = 0, \quad j = 1, 2, \]  
(4.3)
for all \( \tau > \tau_0 \).

We have:

**Theorem 2.** Assume that \( h \) satisfies \( \|h\|_{i^*, \nu, \mu, \sigma} < \infty \) and the conditions (4.1), (4.2) and (4.3). Then
\[ |\phi(y, \tau)| + (1 + |y|)|\nabla \phi(y, \tau)| \lesssim \|h\|_{i^*, \nu, \mu, \sigma} \begin{cases} \frac{1}{\tau^{\nu} \log^{\nu}(1+\rho)} & \rho \leq \sqrt{\tau} \\ \frac{1}{\tau^{\nu-1} \log^{\nu}(1+\rho)} & \rho \geq \sqrt{\tau}. \end{cases} \]

**Proof.** Let
\[ g = \frac{\phi}{U_0} - (-\Delta)^{-1} \phi + c(\tau), \]  
(4.4)
where \( c(\tau) \) is chosen so that
\[ \int_{\mathbb{R}^2} g U_0 = 0. \]  
(4.5)

Note that
\[ \partial_\tau \phi = \nabla \cdot (U_0 \nabla g) + h, \quad \text{in } \mathbb{R}^2 \times (\tau_0, \infty). \]  
(4.6)

We multiply this equation by \( g \) and integrate in \( \mathbb{R}^2 \), using Lemma 3.2:
\[ \frac{1}{2} \partial_\tau \int_{\mathbb{R}^2} \phi g + \int_{\mathbb{R}^2} U_0 |\nabla g|^2 = \int_{\mathbb{R}^2} h g. \]

We use the inequality in Lemma 3.1 to get
\[ \frac{1}{R_1^2} \int_{B_{R_1}} (g - \bar{g}_{R_1})^2 U_0 \leq \int_{\mathbb{R}^2} U_0 |\nabla g|^2 \]
where
\[ \bar{g}_{R_1} = \frac{1}{\int_{B_{R_1}} U_0} \int_{B_{R_1}} g U_0. \]

Here \( R_1 \) is a large positive constant, to be made precise below.

Then
\[ \frac{1}{2} \partial_\tau \int_{\mathbb{R}^2} \phi g + \frac{1}{R_1^2} \int_{B_{R_1}} g^2 U \leq CR_1^2 \int_{\mathbb{R}^2} h^2 U^{-1} + \frac{1}{2R_1^2} \left( \int_{\mathbb{R}^2} g^2 U + C \bar{g}_{R_1} \right). \]

But by (4.5)
\[ \bar{g}_{R_1} = -\frac{1}{\int_{B_{R_1}} U_0} \int_{\mathbb{R}^2 \backslash B_{R_1}} g U_0 \]
so
\[ \bar{g}_{R_1}^2 \leq C \int_{\mathbb{R}^2 \backslash B_{R_1}} g^2 U_0. \]
Therefore
\[
\frac{1}{2} \partial_\tau \int_{\mathbb{R}^2} \phi g + \frac{1}{2R_1^2} \int_{B_{R_1}} g^2 U_0 \lesssim R_1^2 \int_{\mathbb{R}^2} h^2 U_0^{-1} + \frac{1}{R_1^2} \int_{\mathbb{R}^2 \setminus B_{R_1}} g^2 U_0.
\]

We now use Lemma 3.1 to get
\[
\partial_\tau \int_{\mathbb{R}^2} \phi g + \frac{1}{C} \int_{\mathbb{R}^2} \phi g \lesssim R_1^2 \int_{\mathbb{R}^2} h^2 U_0^{-1} + \frac{1}{R_1^2} \int_{\mathbb{R}^2 \setminus B_{R_1}} g^2 U_0.
\]  
(4.7)

Define
\[
A^2 = \sup_{\tau \geq \tau_0} \left\{ \tau^{2\nu} \log^{2 \mu} \tau \int_{J_{\mathbb{R}^2 \setminus B_R}} g^2(t) U_0 \right\}.
\]
Integrating (4.7) and using Lemma 3.1 we find
\[
\int_{\mathbb{R}^2} g^2 U_0 \lesssim \frac{R_1^4 \|h\|_{L^2_{i^*, \nu, \mu, \sigma}}^2}{\tau^{2\nu} \log^{2 \mu} \tau} + \frac{A^2}{\tau^{2\nu} \log^{2 \mu} \tau}.
\]  
(4.8)

Let us use the notation
\[
\|g\|_{L^2(U_0^{1/2})}^2 = \int_{\mathbb{R}^2} g^2 U_0
\]
and we record the estimate (4.8) as
\[
\|g(\tau)\|_{L^2(U_0^{1/2})} \lesssim \frac{M}{\tau^{\nu} \log^{\mu} \tau},
\]
where
\[
M = R_1^4 \|h\|_{L^2_{i^*, \nu, \mu, \sigma}} + A
\]

The idea now is to obtain decay of $g$, and use this decay to show that $A$ can be eliminated from the estimate (4.8).

We define
\[
g_0 = U_0 g
\]
and obtain from (4.6) the equation
\[
\partial_\tau g_0 = U_0 \partial_\tau g = \partial_\tau \phi + U_0 \Delta^{-1} \partial_\tau \phi
\]
\[
= \nabla \cdot (U_0 \nabla g) + h - U_0 (-\Delta)^{-1} \left[ \nabla \cdot (U_0 \nabla g) + h \right]
\]
\[
= \nabla \cdot \left[ U_0 \nabla \left( \frac{g_0}{U_0} \right) \right] + h - U_0 v - U_0 (-\Delta)^{-1} h,
\]  
(4.9)

where
\[
v := (-\Delta)^{-1} \left[ \nabla \cdot (U_0 \nabla g) \right].
\]  
(4.10)

We claim that
\[
|v(y, \tau)| \lesssim \frac{M}{\tau^{\nu} \log^{\mu} \tau (1 + |y|^{2-\epsilon})},
\]  
(4.11)

for any $\epsilon > 0$.

To prove this, we first compute
\[
\nabla \cdot (U_0 \nabla g) = \Delta g U_0 + \nabla U_0 \cdot \nabla g = \Delta (g U_0) - \nabla U_0 \cdot \nabla g - g \Delta U_0,
\]
and hence
\[
v = -g U_0 - (-\Delta)^{-1} \left[ \nabla U_0 \cdot \nabla g + g \Delta U_0 \right].
\]
Let
\[ v_2 = (-\Delta)^{-1} [\nabla U_0 \cdot \nabla g + g \Delta U_0], \quad (4.12) \]
so that
\[ -\Delta v_2 = \nabla U_0 \cdot \nabla g + g \Delta U_0 = \nabla (g \nabla U_0) \quad \text{in } \mathbb{R}^2. \quad (4.13) \]
We claim that for any \( \varepsilon \in (0, 1) \)
\[ |v_2(y, \tau)| \lesssim \frac{M}{\tau^{\nu} \log^\mu \tau (1 + |y|)^{1-\varepsilon}}. \quad (4.14) \]
Indeed, we write equation (4.13) on the sphere \( S^2 \)
\[ -\Delta_{S^2} \tilde{v}_2 = \nabla_{S^2} (\tilde{g} \nabla_{S^2} \tilde{U}_0) \quad \text{in } S^2, \quad (4.15) \]
where \( \tilde{v}_2 = v_2 \circ \Pi, \tilde{g} = g \circ \Pi, \tilde{U}_0 = U_0 \circ \Pi, \) and \( \Pi \) is the stereographic projection defined in section ??.

We note that the solution of the previous equation is defined up to an additive constant. In \( \tilde{v}_2 \) this constant is fixed by the condition \( \tilde{v}_2(P) = 0 \), which corresponds to the solution selected by the formula (4.12).

Observe that
\[ \int_{S^2} \tilde{g}^2 |\nabla_{S^2} \tilde{U}_0|^2 \lesssim \int_{\mathbb{R}^2} g^2 |\nabla \tilde{U}_0|^2 \lesssim \|g\|_{L^2(U_0^{1/2})}^2. \]
Using standard elliptic theory we find that \( \tilde{v}_2 \in H^1(S^2) \) and \( \|\tilde{v}_2\|_{H^1} \lesssim \|g\|_{L^2(U_0^{1/2})}. \)

We deduce that \( \|\tilde{v}_2\|_{C^{\alpha}(S^2)} \lesssim \|g\|_{L^2(U_0^{1/2})} \) for any \( \alpha \in (0, 1). \) From this we get (4.14).

With a similar argument we get that
\[ |(-\Delta)^{-1} h(y, \tau)| \lesssim \frac{\|h\|_{L^2(U_0^{1/2})}}{\tau^{\nu} \log^\mu \tau (1 + |y|)^{1-\varepsilon}}, \quad (4.16) \]
Now we claim that
\[ |g_0(y, \tau)| \lesssim \frac{M}{\tau^{\nu} \log^\mu \tau (1 + |y|)^2}. \quad (4.17) \]
Indeed, we write (4.9) as
\[ \partial_\tau g_0 = \Delta g_0 - \nabla g_0 \nabla V_0 + h + 2U_0 g_0 + U_0 v_2 - U_0 (-\Delta)^{-1} h. \quad (4.18) \]
Consider a point \( y \in \mathbb{R}^2. \) From
\[ \|g\|_{L^2(B_1(y))} \lesssim (1 + |y|)^2 \|g(\tau)\|_{L^2(U_0^{1/2})}, \]
we see that
\[ \|g_0\|_{L^2(B_1(y))} \lesssim \frac{M}{\tau^{\nu} \log^\mu \tau (1 + |y|)^2}. \quad (4.19) \]
Applying standard parabolic estimates to (4.18) restricted to \( B_1(y) \times (\tau, \tau + 1), \) together with (4.19), (4.14), (4.16) we obtain (4.17). From this inequality and a scaling combined with parabolic estimates we also obtain
\[ |
abla g_0(y, \tau)| \lesssim \frac{M}{\tau^{\nu} \log^\mu \tau (1 + |y|)^2}. \quad (4.20) \]
We reconsider now (4.15) and observe that
\[
\operatorname{div}_{S^2}(\tilde{g} \nabla_{S^2} \tilde{U}_0) = \nabla_{S^2} \tilde{g} \nabla_{S^2} \tilde{U}_0 + \tilde{g} \Delta_{S^2} \tilde{U}_0
\]
\[
= \frac{(1 + |y|^2)^2}{4} [\nabla_{R^2} g \nabla_{R^2} U_0 + g \Delta_{R^2} U_0].
\]
Using (4.17), (4.20) and \(g_0 = gU_0\) we get that
\[
|\operatorname{div}_{S^2}(\tilde{g} \nabla_{S^2} \tilde{U}_0)| \lesssim \frac{M}{\tau^\nu \log^{\alpha} \tau}.
\]
Using standard elliptic regularity we conclude that \(\tilde{v}_2 \in C^{1,\alpha}(S^2)\) for any \(\alpha \in (0, 1)\) and the estimate
\[
||\tilde{v}_2||_{C^{1,\alpha}(S^2)} \lesssim \frac{M}{\tau^\nu \log^{\alpha} \tau}.
\]
Since \(\tilde{v}_2(P) = 0\), from a Taylor expansion of \(\tilde{v}_2\) about \(P\) we obtain for the original \(v_2\) the expansions
\[
\begin{cases}
|v_2(y, \tau) - a(\tau) \cdot y| \lesssim \frac{M}{\tau^\nu \log^{\alpha} \tau (1 + |y|)^{1+\alpha}}, \\
|\nabla v_2(y, \tau) - a(\tau) \cdot y| \lesssim \frac{M}{\tau^\nu \log^{\alpha} \tau (1 + |y|)^{2+\alpha}}.
\end{cases}
\] (4.21)
for some \(a(\tau) = (a_1(\tau), a_2(\tau))\), for any \(\alpha \in (0, 1)\). By the definition of \(v\) (4.22), the estimate for \(g_0\) (4.17), (4.20), and the expansion (4.21) we obtain also for \(v\):
\[
\begin{cases}
|v(y, \tau) - a(\tau) \cdot y| \lesssim \frac{M}{\tau^\nu \log^{\alpha} \tau (1 + |y|)^{1+\alpha}}, \\
|\nabla v(y, \tau) - a(\tau) \cdot y| \lesssim \frac{M}{\tau^\nu \log^{\alpha} \tau (1 + |y|)^{2+\alpha}}.
\end{cases}
\] (4.22)
We will show next that actually \(a_1(\tau) = a_2(\tau) = 0\). For this we use equation (4.22), and we multiply by \(y_i\) and integrate by parts. First we observe first that for \(i = 1, 2\)
\[
\int_{R^2} \nabla(U_0 \nabla g) y_i \, dy = 0.
\] (4.23)
Indeed,
\[
\int_{R^2} \nabla(U_0 \nabla g) y_i \, dy = - \int_{R^2} U_0 \nabla g e_i = \int_{R^2} g \nabla U_0 e_i.
\]
But from (4.4), letting \(\psi_0 = (\Delta)^{-1} \phi\) and \(\psi = \psi_0 - c(\tau)\) we have
\[
-\Delta \psi - U_0 \psi = U_0 g.
\]
Multiplying this equation by \(z_i = \nabla V_0 e_i\) defined in (3.2) and integrating we get
\[
\int_{R^2} g U_0 \nabla V_0 e_i = 0,
\]
which is the desired claim (4.23). We note that the integrations by parts are justified by the decay
\[
|\psi_0(y, \tau)| + (1 + |y|)|\nabla \psi_0(y, \tau)| \lesssim C(\tau) \frac{1}{(1 + |y|)^{\sigma}}.
\]
Now we multiply the equation by $y_1$ and integrate in a ball $B_R(0)$, where $R > 0$ and later we let $R \to \infty$. Integrating we get
\[ \int_{\partial B_R} (-\partial_y y_1 + v) = \int_{B_R} \nabla(U_0 \nabla) y_1 \, dy \]

Using polar coordinates $y = \rho e^{i\theta}$ and (4.22), we see that
\[ \int_{\partial B_R} (-\partial_y y_1 + v) = 2\pi a_1(\tau) + O(R^{-\alpha}). \]

Letting $R \to \infty$ and using (4.23) we conclude that $a_1(\tau) = 0$. Similarly $a_2(\tau) = 0$.

We deduce from this and (4.22) that
\[ |v(y, \tau)| \lesssim M \frac{\rho^{2}}{\tau^{\nu} \log^{\mu} (1 + |y|)^{1+\alpha}}. \]

This is the desired conclusion (4.11).

A similar proof, using (4.3) yields
\[ |(-\Delta)^{-1} h(y, \tau)| \lesssim \frac{||h||_{i,*,\nu,\mu,\sigma}}{\tau^{\nu} \log^{\mu} (1 + |y|)^{1+\alpha}}. \] (4.24)

Now we choose a large constant $R_0$ so that we can use the maximum principle for the parabolic operator $\partial_{\tau} f - \nabla \cdot [U_0 \nabla f]$ is valid outside the ball $B_{R_0}(0)$. Indeed, we have
\[ \nabla \cdot \left( U_0 \nabla \left( \frac{f}{U_0} \right) \right) = \Delta f - \nabla V_0 \nabla f + U_0 f = \partial_{\rho\rho} f + \frac{5}{\rho} \partial_{\theta} f + \frac{1}{\rho^2} \partial_{\theta\theta} f + Df \]
where $Df = O(\frac{1}{\rho}) \partial_{\rho} f + O(\frac{1}{\rho^2}) f$ represent lower order terms. Using the maximum principle and an appropriate barrier in $\mathbb{R}^2 \setminus B_{R_0}$, as constructed in Theorem 3 below, we get that

\[ |g_0(y, \tau)| \lesssim \frac{R^2 ||h||_{i,*,\nu,\mu,\sigma} + A}{\tau^{\nu} \log^{\mu} (1 + |y|)^{3+\sigma}}, \quad |y| \leq \sqrt{\tau} \]

and

\[ |g_0(y, \tau)| \lesssim \frac{R^2 ||h||_{i,*,\nu,\mu,\sigma} + A}{\tau^{\nu-1} \log^{\mu} (1 + |y|)^{3+\sigma}}, \quad |y| \geq \sqrt{\tau}. \]

We use this estimate to compute
\[ \int_{\mathbb{R}^2 \setminus B_{R_1}} g^2 U = \int_{\mathbb{R}^2 \setminus B_{R_1}} g^2 U^{-1} \lesssim \frac{1}{R_1^\alpha} \frac{R^4 ||h||_{i,*,\nu,\mu,\sigma}^2 + A^2}{\tau^{2\nu} \log^{2\mu} \tau}. \]

This implies that
\[ A^2 \leq C \frac{1}{R_1^\alpha} (R_1^4 ||h||_{i,*,\nu,\mu,\sigma}^2 + A^2), \]

where $C$ is a constant from previous inequalities, which is independent of $R_1$.

Choosing a fixed $R_1$ large then implies that
\[ A^2 \lesssim R_1^{4-2\sigma} ||h||_{i,*,\nu,\mu,\sigma}^2. \]
We then conclude that
\[
|g_0(y, t)| \lesssim \|h\|_{\ast, \nu, \mu, \sigma} \begin{cases} 
\frac{1}{\tau^\nu \log^{\nu} \tau (1 + |y|)^{1+\sigma}}, & |y| \leq \sqrt{\tau} \\
\frac{1}{\tau^{\nu-1} \log^{\nu} \tau (1 + |y|)^{1+\sigma}}, & |y| \geq \sqrt{\tau}.
\end{cases}
\]
From parabolic estimates we also find
\[
|\nabla g_0(y, t)| \lesssim \|h\|_{\ast, \nu, \mu, \sigma} \begin{cases} 
\frac{1}{\tau^\nu \log^{\nu} \tau (1 + |y|)^{1+\sigma}}, & |y| \leq \sqrt{\tau} \\
\frac{1}{\tau^{\nu-1} \log^{\nu} \tau (1 + |y|)^{1+\sigma}}, & |y| \geq \sqrt{\tau}.
\end{cases}
\] (4.25)

Now we estimate \( \phi \). We decompose
\[
\phi = \phi^\perp + \omega(\tau) Z_0,
\]
where \( Z_0 \) is defined in (2.26). We then have
\[
g = \frac{\phi^\perp}{U_0} - (\Delta^{-1}) \phi^\perp + c(t).
\]
We let \( \psi = (-\Delta^{-1}) \phi^\perp \) and see that
\[
g U = \Delta \psi + U_0 \psi.
\]
Integrating the equation times \( |y|^2 \) we get
\[
\int_{\mathbb{R}^2} \phi(y, \tau) |y|^2 \, dy = 0, \quad \forall \tau > \tau_0,
\]
and this is equivalent to
\[
\int_{\mathbb{R}^2} g Z_0 = 0, \quad \forall \tau > \tau_0,
\]
where \( Z_0 \) is defined in (2.26). We then can solve the equation for \( \psi \) and find
\[
|\psi(y, \tau)| \lesssim \|h\|_{\ast, \nu, \mu, \sigma} \begin{cases} 
\frac{1}{\tau^\nu \log^{\nu} \tau (1 + |y|)^{1+\sigma}}, & |y| \leq \sqrt{\tau} \\
\frac{1}{\tau^{\nu-1} \log^{\nu} \tau (1 + |y|)^{1+\sigma}}, & |y| \geq \sqrt{\tau}.
\end{cases}
\]
Since
\[
\phi^\perp = U_0 (g - \psi)
\]
we find that
\[
|\phi^\perp(y, \tau)| \lesssim \|h\|_{\ast, \nu, \mu, \sigma} \begin{cases} 
\frac{1}{\tau^\nu \log^{\nu} \tau (1 + |y|)^{1+\sigma}}, & |y| \leq \sqrt{\tau} \\
\frac{1}{\tau^{\nu-1} \log^{\nu} \tau (1 + |y|)^{1+\sigma}}, & |y| \geq \sqrt{\tau}.
\end{cases}
\] (4.26)

Finally we estimate \( \omega(\tau) \). We have
\[
\partial_\tau \phi^\perp + \omega \cdot z = L[\phi] + h.
\]
We multiply by \( |y|^2 \) and integrate in \( B_{R_2} \) where \( R_2 \to \infty \) and in a time interval \([\tau_1, \tau_2]\). We get
\[
\int_{B_{R_2}} (\phi(\tau_2)^\perp - \phi(\tau_1)^\perp) |y|^2 \, dy + (\omega(\tau_2) - \omega(\tau_1)) \int_{B_{R_2}} Z_0 |y|^2 \, dy = \int_{\tau_1}^{\tau_2} \int_{B_{R_2}} L[\phi] |y|^2 \, dy \, d\tau.
\]
Let us observe that if \( R_2 \geq \sqrt{\tau} \) then
\[
\int_{B_{R_2}} |\phi(y, \tau)^\perp| |y|^2 \, dy \lesssim \frac{1}{\tau^{\nu-1} \log^{\nu} \tau R_2^{1+\sigma}}.
\]
On the other hand
\[ \int_{B_{R_2^2}} L[\phi]|y|^2 \, dy = \int_{B_{R_2^2}} gZ_0 \, dy + \int_{\partial B_{R_2^2}} U_0 |y|^2 \nabla g \cdot \nu - \int_{\partial B_{R_2^2}} gU_0 y \cdot \nu, \]
and
\[ \left| \int_{B_{R_2^2}} gZ_0 \, dy \right| \leq \int_{B_{R_2^2}} |g_0| \, dy \lesssim \frac{1}{\tau^\mu \log^\mu \tau}. \]

\[ \left| \int_{\partial B_{R_2^2}} U_0 |y|^2 \nabla g \cdot \nu \right| + \left| \int_{\partial B_{R_2^2}} gU_0 y \cdot \nu \right| \lesssim \frac{1}{\tau^\nu \log^\nu \tau R_2^\nu}. \]

Since
\[ \int_{B_{R_2^2}} Z_0 |y|^2 \, dy \sim \log R_2, \]
letting \( R_2 \to \infty \) we find that \( \omega(\tau_2) = \omega(\tau_1) \). Hence \( \omega \equiv const \) and since we start with \( \omega(0) = 0 \) we deduce \( \omega \equiv 0 \). Hence the estimate (4.26) gives the desired estimate for \( \phi \). The estimate for the gradient of \( \phi \) comes from the corresponding estimate for the gradient of \( \phi^\perp \), which is obtained similarly from (4.25). \( \square \)

5. Outer problem

**Theorem 3.** Let us consider the equation
\[ \partial_t \phi^o = L^o[\phi^o] + h(x, t), \]
where
\[ |h(x, t)| \leq \frac{\lambda^{-2}}{t^a \log^\mu t (\frac{|x-x_1|}{\lambda} + 1)^b}. \]

If \( 2 < b < 6 \) and \( 1 < a < 3 \), then
\[ |\phi^o(x, t)| \lesssim \min \left( \frac{1}{t^a \log^\mu t (\frac{|x-x_1|}{\lambda} + 1)^b}, \frac{\lambda^{-2}}{t^{a-1} \log^\mu t (\frac{|x-x_1|}{\lambda} + 1)^b} \right). \]

**Proof.** We solve
\[ \partial_t \varphi = \partial_{\rho \rho} \varphi + \frac{5}{\rho} \partial_\rho \varphi + \tilde{h}(\rho, t) \]
Define
\[ \rho = \sqrt{r^2 + \lambda^2}, \]
\[ \tilde{\varphi}(r, t) = \varphi(\sqrt{r^2 + \lambda^2}, t). \]
We compute
\[
\begin{align*}
\partial_t \bar{\varphi} &= \partial_t \varphi + \partial_\rho \varphi \frac{\lambda \dot{\lambda}}{\sqrt{r^2 + \lambda^2}} \\
\partial_r \bar{\varphi} &= \partial_\rho \varphi \frac{r}{\sqrt{r^2 + \lambda^2}} \\
\partial_{rr} \bar{\varphi} &= \partial_{pp} \varphi \frac{r^2}{r^2 + \lambda^2} + \partial_\rho \varphi \left( \frac{1}{\sqrt{r^2 + \lambda^2}} - \frac{r^2}{(r^2 + \lambda^2)^{\frac{3}{2}}} \right) \\
&= \partial_{pp} \varphi \frac{r^2}{r^2 + \lambda^2} + \partial_\rho \varphi \frac{\lambda^2}{(r^2 + \lambda^2)^{\frac{3}{2}}}
\end{align*}
\]

Then
\[
- \partial_t \bar{\varphi} + \partial_{rr} \bar{\varphi} + \left( \frac{1}{r} + \frac{4r}{r^2 + \lambda^2} \right) \partial_r \bar{\varphi}
\]
\[
= - \partial_t \varphi - \partial_\rho \varphi \frac{\lambda \dot{\lambda}}{\sqrt{r^2 + \lambda^2}} + \partial_{pp} \varphi \frac{r^2}{r^2 + \lambda^2} + \partial_\rho \varphi \frac{\lambda^2}{(r^2 + \lambda^2)^{\frac{3}{2}}}
\]
\[
+ \partial_\rho \varphi \frac{1}{r^2 + \lambda^2} \left( \frac{1}{r} + \frac{4r}{r^2 + \lambda^2} \right)
\]
\[
= - \partial_t \varphi + \partial_{pp} \varphi + \frac{5}{\rho} \partial_\rho \varphi - \partial_{pp} \varphi \frac{\lambda^2}{\rho^2} + \partial_\rho \varphi \left( - \frac{\lambda \dot{\lambda}}{\rho} + \frac{\lambda^2}{\rho^3} + \frac{1}{\rho} + \frac{4r^2}{\rho^3} - \frac{5}{\rho} \right)
\]
\[
= - \partial_t \varphi + \partial_{pp} \varphi + \frac{5}{\rho} \partial_\rho \varphi - \partial_{pp} \varphi \frac{\lambda^2}{\rho^2} + \partial_\rho \varphi \left( - \frac{\lambda \dot{\lambda}}{\rho} - \frac{3 \lambda^2}{\rho^3} \right)
\]

Therefore $\bar{\varphi}$ serves as a supersolution. Then we take a cut-off function $\eta(x, t) = \chi_0(\frac{r}{\sqrt{t}})$. Using an appropriate self-similar solution $\psi$ we find a supersolution for the problem of the form $\eta \bar{\varphi} + \psi$.

\[\square\]

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