AXIALLY SYMMETRIC SOLUTIONS OF ALLEN-CAHN EQUATION WITH FINITE MORSE INDEX∗

CHANGFENG GUI†, KELEI WANG‡ AND JUNCHENG WEI§

Abstract. In this paper we study axially symmetric solutions of Allen-Cahn equation with finite Morse index. It is shown that there does not exist such a solution in dimensions between 4 and 10. In dimension 3, we prove that these solutions have finitely many ends. Furthermore, the solution has exactly two ends if its Morse index equals 1.

1. Introduction

In this paper we study axially symmetric solutions of the Allen-Cahn equation
\[ \Delta u = W'(u), \quad \text{in } \mathbb{R}^{n+1}. \]
Here \( W(u) \) is a general double well potential, that is, \( W \in C^4([-1,1]) \) satisfying
- \( W > 0 \) in \((-1,1)\) and \( W(\pm 1) = 0 \);
- \( W'(\pm 1) = 0 \) and \( W''(-1) = W''(1) = 2 \);
- \( W \) is even and 0 is the unique critical point of \( W \) in \((-1,1)\).

A typical model is given by \( W(u) = (1 - u^2)^2/4 \).

For this class of double well potential, there exists a unique solution to the following one dimensional problem
\[ g''(t) = W'(g(t)), \quad g(0) = 0 \quad \text{and} \quad \lim_{t \to \pm \infty} g(t) = \pm 1. \]
Moreover, as \( t \to \pm \infty, g(t) \) converges exponentially to \( \pm 1 \) and the following quantity is well defined
\[ \sigma_0 := \int_{-\infty}^{+\infty} \left[ \frac{1}{2} g'(t)^2 + W(g(t)) \right] dt \in (0, +\infty). \]

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In fact, as $t \to \pm \infty$, the following expansions hold: there exists a positive constant $A$ such that for all $|t|$ large,

$$
\begin{align*}
g(t) &= (1 - Ae^{-\sqrt{2}|t|})\text{sgn}(t) + O(e^{-2\sqrt{2}|t|}), \\
g'(t) &= \sqrt{2}Ae^{-\sqrt{2}|t|} + O(e^{-2\sqrt{2}|t|}), \\
g''(t) &= -2Ae^{-\sqrt{2}|t|} + O(e^{-2\sqrt{2}|t|}).
\end{align*}
$$

Denote points in $\mathbb{R}^{n+1}$ by $(x_1, \cdots, x_n, z)$ and let $r := \sqrt{x_1^2 + \cdots + x_n^2}$.

**Definition 1.1.**

- A function $u$ is axially symmetric if $u(x_1, \cdots, x_n, z) = u(r, z)$.
- A solution of (1.1) is stable in a domain $\Omega \subset \mathbb{R}^{n+1}$ if for any $\varphi \in C_0^{\infty}(\Omega)$,
  $$Q_\Omega(\varphi) := \int_\Omega [|
abla \varphi|^2 + W''(u)\varphi^2] \geq 0.$$  
- A solution of (1.1) has finite Morse index in $\mathbb{R}^{n+1}$ if
  $$\sup_{R>0} \dim \{X \subset C_0^{\infty}(B_R^{n+1}(0)) : Q|_X < 0\} < +\infty.$$  

It is well known that the finite Morse index condition is equivalent to the condition of being stable outside a compact set, see [6].

**Definition 1.2.** An axially symmetric solution of (1.1) has finitely many ends if for some $R > 0$,

- $u \neq 0$ in $B_R^n(0) \times \{|z| > R\}$;
- outside $\mathcal{C}_R := B_R^n(0) \times \mathbb{R}$, $\{u = 0\}$ consists of finitely many graphs $\Gamma_\alpha$, where
  $$\Gamma_\alpha = \{z = f_\alpha(r)\}, \quad \alpha = 1, \cdots, Q,$$
  and $f_1 < \cdots < f_Q$.

Our first main result is

**Theorem 1.3.** If $3 \leq n \leq 9$, any axially symmetric solution of (1.1), which is stable outside a cylinder $\mathcal{C}_R$, depends only on $z$.

In other words, the solution has exactly one end or it is one dimensional, i.e. all of its level sets are hyperplanes of the form $\{z = t\}$. Therefore for $3 \leq n \leq 9$, there does not exist axially symmetric solutions which is stable outside a cylinder, except the trivial ones (i.e., constant solutions $\pm 1$ and $g$ in (1.2)).

The dimension bound in this theorem is sharp. On one hand, if $n \geq 10$, there do exist stable, axially symmetric solutions of (1.1) in $\mathbb{R}^{n+1}$ with two ends, see Agudelo-Del Pino-Wei [1]. (The two-end solutions constructed in this paper for $3 \leq n \leq 9$ are also shown to be unstable by a different argument. Our proof of Theorem 1.3 will rely on an idea of Dancer and Farina [4].) On the other hand, nontrivial axially symmetric solutions with finite Morse index in $\mathbb{R}^3$ also exist. (See del Pino-Kowalczyk-Wei [5].) However we show that
Theorem 1.4. If \( n = 2 \), an axially symmetric solution of (1.1) with finite Morse index has finitely many ends. Moreover, there exists a constant \( C \) such that for any \( x \in \mathbb{R}^3 \) and \( R > 0 \),

\[
\int_{B_R(x)} \left[ \frac{1}{2} |\nabla u|^2 + W(u) \right] \leq CR^2.
\]

Concerning solutions with a low Morse index we first show that

Theorem 1.5. If \( n = 2 \), any axially symmetric, stable solution of (1.1) depends only on \( z \).

Next we prove that

Theorem 1.6. Any axially symmetric solution of (1.1) with Morse index 1 in \( \mathbb{R}^3 \) has exactly two ends.

Two end solutions in \( \mathbb{R}^3 \) have been studied in detail in Gui-Liu-Wei [9]. They showed that for each \( k \in (\sqrt{2}, +\infty) \) there exist two-ended axially symmetric solutions whose zero level sets approximately look like \( \{ z = \pm k \log r \} \). Parallel to Schoen’s result in minimal surfaces [11], one may ask the following natural question:

Conjecture: All two-ended solutions to Allen-Cahn equation in \( \mathbb{R}^3 \) must be axially symmetric.

We introduce some notations used in the proof of Theorems 1.3-1.6. Taking \((r, z)\) as coordinates in the plane, after an even extension to \( \{ r < 0 \} \), an axially symmetric function \( u \) can be viewed as a smooth function defined on \( \mathbb{R}^2 \). Now (1.1) is written as

\[
u_{rr} + \frac{n-1}{r} u_r + u_{zz} = W'(u).
\]

We use subscripts to denote differentiation, e.g. \( u_z := \frac{\partial u}{\partial z} \). A nodal domain of \( u_z \) is a connected component of \( \{ u_z \neq 0 \} \). Sometimes we will identify various objects in \( \mathbb{R}^{n+1} \) with the corresponding ones in the \((r, z)\)-plane, if they have axial symmetry.

To prove Theorems 1.3-1.6 we follow from a strategy used by the second and the third authors [18]. One of the main difficulties is the possibility of an infinite tree of nodal domains of \( u_z \). Here we explore the decaying properties of the curvature to exclude this scenario.

The remaining part of this paper is organized as follows. In Section 2 we give a curvature decay estimate on level sets of \( u \). This curvature estimate allows us to determine the topology and geometry of ends in Section 3. In Section 4 we show that interaction between different ends is modeled by a Toda system. The case \( 3 \leq n \leq 9 \) is analysed in Section 5, while Section 6 is devoted to the proof of the \( n = 2 \) case. Finally, Theorem 1.5 and Theorem 1.6 are proved in Section 7.
2. Curvature decay

In this section $u$ denotes an axially symmetric solution in $\mathbb{R}^{n+1}$, which is stable outside a cylinder $C_R$. We will establish a technical result on curvature decay of level sets of $u$.

Let us first recall several results on stable solutions of (1.1). By [12], given a domain $\Omega \subset \mathbb{R}^{n+1}$, the stability of $u$ in $\Omega$ is equivalent to the following Sternberg-Zumbrun inequality

$$\int_{\Omega} |\nabla \varphi|^2 |\nabla u|^2 \geq \int_{\Omega} \varphi^2 |B(u)|^2 |\nabla u|^2, \quad \forall \varphi \in C_0^\infty(\Omega).$$

In the above,

$$|B(u)|^2 := \frac{|\nabla^2 u|^2 - |\nabla |\nabla u||^2}{|\nabla u|^2} = |A|^2 + |\nabla T \log |\nabla u||^2,$$

where $A$ is the second fundamental form of the level set of $u$ and $\nabla T$ is the tangential derivative along the level set.

The following Stable De Giorgi theorem in dimension 2 is well known, see [8].

**Theorem 2.1.** Suppose $u$ is a stable solution of (1.1) in $\mathbb{R}^2$. Then $u$ is one dimensional. In particular, $|B(u)|^2 \equiv 0$.

Using this theorem we show that away from the $z$-axis, $u$ looks like an one dimensional solution at $O(1)$ scales.

**Proposition 2.2.** Suppose $u$ is an axially symmetric solution of (1.4) in $\mathbb{R}^{n+1}$, which is stable outside a cylinder $C_R$. Then for any $\varepsilon > 0$, there exists an $R(\varepsilon) > R$ such that for any $r \geq R(\varepsilon)$ and $z \in \mathbb{R}$ where $u(r, z) = 0$, we have

$$\|u - g\|_{C^2(B_{R^2+1}(r, z))} \leq \varepsilon.$$

Here, by abusing notations, $g$ denotes a one dimensional solution in the $(r, z)$-plane.

**Proof.** Take an arbitrary sequence $R_i \to +\infty$ and $z_i \in \mathbb{R}$ with $u(R_i, z_i) = 0$. We need to show that, after passing to a subsequence, $u_i(r, z) := u(R_i + r, z_i + z)$ converges to a one dimensional solution of (1.1) in $C^2_{loc}(\mathbb{R}^2)$.

By standard elliptic estimates we may assume $u_i$ converge to $u_\infty$ in $C^2_{loc}(\mathbb{R}^2)$. Passing to the limit in (1.4) we see $u_\infty$ is a solution of (1.1) in $\mathbb{R}^2$.

Because $u$ is axially symmetric and stable outside $C_R$, there exists an axially symmetric function $\varphi$, which is positive outside $C_R$, such that

$$\varphi_{rr} + \frac{n-1}{r} \varphi_r + \varphi_{zz} = W''(u) \varphi, \quad \text{outside} \quad C_R.$$

Define

$$\varphi^i(r, z) := \frac{1}{\varphi(R_i, z_i)} \varphi(R_i + r, z_i + z).$$

For any $R > 0$, it satisfies

$$\varphi^i_{rr} + \frac{n-1}{R_i + r} \varphi^i_r + \varphi^i_{zz} = W''(u_i) \varphi^i, \quad \text{in} \quad B_R(0).$$
By definition, \( \varphi'(0) = 1 \) and \( \varphi^i > 0 \). Then by Harnack inequality and standard elliptic estimates, after passing to a subsequence we may take a limit \( \varphi^i \to \varphi^\infty \) in \( C^2_{\text{loc}}(\mathbb{R}^2) \). Here \( \varphi^\infty \) satisfies

\[
\varphi^\infty_{rr} + \varphi^\infty_{zz} = W''(u^\infty)\varphi^\infty, \quad \varphi^\infty > 0 \quad \text{in} \quad \mathbb{R}^2.
\]

Hence \( u^\infty \) is a stable solution of (1.1) in \( \mathbb{R}^2 \). By Theorem 2.1, \( u^\infty \) is one dimensional. \( \square \)

**Corollary 2.3.** Suppose \( u \) is an axially symmetric solution of (1.4) in \( \mathbb{R}^{n+1} \), which is stable outside a cylinder \( C_R \). For any \( b \in (0, 1) \), there exists a constant \( C(b) \) such that in \( \{|u| < 1 - b\} \setminus C_{R(b)} \),

\[
|B(u)(r, z)| \to 0.
\]

The main technical tool we need in this paper is the following decay estimate on \( |B(u)|^2 \).

**Theorem 2.4.** Suppose \( u \) is an axially symmetric solution of (1.4) in \( \mathbb{R}^{n+1} \), which is stable outside a cylinder \( C_R \). For any \( b \in (0, 1) \), there exists a constant \( C(b) \) such that in \( \{|u| < 1 - b\} \setminus C_{R(b)} \),

\[
|B(u)(r, z)|^2 \leq C(b)r^{-2}
\]

and

\[
|H(u)(r, z)| \leq C(b)r^{-2}.
\]

In the above \( H(u)(r, z) \) denotes the mean curvature of the level set \( \{u = u(r, z)\} \) at the point \( (r, z) \). The proof of this theorem is similar to the two dimensional case in [18]. By a blow up method, it is reduced to the second order estimate established in [17]. Note that here no condition on the dimension \( n \) is needed, because as in the proof of Proposition 2.2, the limiting problem after blow up is essentially a two dimensional problem and then the estimate in [17] is applicable.

### 3. Geometry of ends

In this section \( u \) denotes an axially symmetric solution of (1.4) in \( \mathbb{R}^{n+1} \), \( n \geq 2 \), which is stable outside a cylinder \( C_R \). Here and henceforth, a small constant \( b \in (0, 1) \) will be fixed. Notations introduced in the previous section will be kept, too. Take a constant \( R_1 > R(b) \) so that it satisfies

\[
C(b)R_1^{-2} < R_1^{-1}.
\]

By Theorem 2.4, \( \{0\} \setminus C_{R_1} = \bigcup_{\alpha \in A} \Gamma_\alpha \) for an index set \( A \). For each \( \alpha, \Gamma_\alpha \) is a connected smooth embedded hypersurface with or without boundary. Furthermore, \( \Gamma_\alpha \cap \Gamma_\beta = \emptyset \) if \( \alpha \neq \beta \). Finally, since \( u \) is axially symmetric, for each \( \alpha \in A \), \( \Gamma_\alpha \) is also axially symmetric. As a consequence, \( \Gamma_\alpha \) is identified with a smooth curve in the \( (r, z) \) plane.

Viewing \( \Gamma_\alpha \) as a smooth curve in the \( (r, z) \) plane and \( r \) as a function defined on \( \Gamma_\alpha \), we have

**Lemma 3.1.** Every critical point of \( r \) in the interior of \( \Gamma_\alpha \) is a strict local minima.
Proof. Assume by the contrary, there exists a point \((r_*, z_*)\) in the interior of one \(\Gamma_\alpha\), which is a critical point of \(r\) but not a strict local minima. By Proposition 2.2 and Corollary 2.3, in a neighborhood of \((r_*, z_*)\), \(\Gamma_\alpha = \{ r = f_\alpha(z) \}\). By our assumptions, \(f_\alpha(z_*) = r_*\), \(f'_\alpha(z_*) = 0\) and \(f''_\alpha(z_*) \leq 0\). Hence

\[
H_{\Gamma_\alpha}(r_*, z_*) \geq \frac{1}{r_*}.
\]

In view of (3.1), this is a contradiction with Theorem 2.4. \(\square\)

Since \(\Gamma_\alpha\) is a connected smooth curve with end points (if there are) in \(\partial \mathcal{C}_{R_1}\), by this lemma we see there is no local maxima and at most one local minima of \(r\) in the interior of \(\Gamma_\alpha\). There are two cases:

**Type I.** \(\Gamma_\alpha\) is diffeomorphic to \([0, +\infty)\) and it has exactly one end point on \(\partial \mathcal{C}_{R_1}\);

**Type II.** \(\Gamma_\alpha\) is diffeomorphic to \((-\infty, +\infty)\) and its boundary is empty.

If \(\Gamma_\alpha\) is of type I, \(r\) is a strictly increasing function with respect to a parametrization of \(\Gamma_\alpha\). Hence it can be represented by the graph \(\{ z = f_\alpha(r) \}\), where \(f_\alpha \in C^1([R_1, +\infty))\). (Higher order regularity on \(f_\alpha\) follows by applying the implicit function theorem to \(u\).)

If \(\Gamma_\alpha\) is of type II, there exists a point \((R_\alpha, z_\alpha)\), which is the unique minima of \(r\) on \(\Gamma_\alpha\). As in Type I case, \(\Gamma_\alpha \setminus \{(R_\alpha, z_\alpha)\} = \Gamma_\alpha^+ \cup \Gamma_\alpha^-\), where \(\Gamma_\alpha^\pm\) can be represented by two graphs \(\{ z = f_\alpha^\pm(r) \}\). Here \(f_\alpha^+ > f_\alpha^-\) on \((R_\alpha, +\infty)\) and \(f_\alpha^+(R_\alpha) = f_\alpha^-(R_\alpha) = z_\alpha\).

**Proposition 3.2.** There exists a constant \(R_2 > R_1\) such that for any type II end \(\Gamma_\alpha\), it holds that \(R_\alpha < R_2\).

**Proof.** Assume by the contrary, there exists a sequence of type II ends \(\Gamma_k\) such that \(R_k \to +\infty\).

By Theorem 2.4, the rescalings \(\Sigma_k := R_k^{-1}[\Gamma_k - (0, z_k)]\) have uniformly bounded curvatures and their mean curvatures converge to 0 uniformly. By standard elliptic estimates, after passing to a subsequence of \(k\), \(\Sigma_k\) converges smoothly to an axially symmetric, smooth minimal hypersurface \(\Sigma_\infty\). Moreover, there exist two functions \(f_\infty^\pm \in C^2((1, +\infty))\) such that

\[
\Sigma_\infty \setminus \{(1, 0)\} = \{(r, z) : z = f_\infty^\pm(r)\}.
\]

Hence \(\Sigma_\infty\) is the standard catenoid. By [13], it is unstable. (Indeed, its Morse index is exactly 1.)

On the other hand, we claim that \(\Sigma_\infty\) inherits the stability from \(u\), thus arriving at a contradiction. To this end, let \(u_k(r, z) := u(R_k r, R_k(z_k + z))\). It is a solution of the singularly perturbed Allen-Cahn equation

\[
\Delta u_k = R_k^2 W'(u_k).
\]

Since \(u\) is stable outside \(\mathcal{C}_{R_1}\), \(u_k\) is stable outside \(\mathcal{C}_{R_k/R_k}\). Note that \(\Sigma_k\) is a connected component of \(\{ u_k = 0 \}\) and it is totally located outside \(\mathcal{C}_1\). Next we divide the discussion into two cases.

- Suppose there exists another connected component of \(\{ u_k = 0 \}\), denoted by \(\overline{\Sigma}_k\), also converging to \(\Sigma_\infty\) in a ball \(B_r(p)\) for some \(r > 0\) and \(p \in \Sigma_\infty\).
By Theorem 2.4, $\tilde{\Sigma}_k$ enjoys the same regularity as for $\Sigma_k$. Hence by the axial symmetry of $\tilde{\Sigma}_k$ and the uniqueness of catenoid, $\tilde{\Sigma}_k$ converges to $\Sigma_\infty$ everywhere. In this case we can construct a positive Jacobi field on $\Sigma_\infty$ as in [3, Theorem 4.1], which implies the stability of $\Sigma_\infty$

- Suppose there is only one such a component in a fixed neighborhood $\mathcal{N}$ of $\Sigma_\infty$. Since $\Sigma_\infty \subset \{ r \geq 1 \}$, we can take $\mathcal{N} \subset \{ r > 1/2 \}$. Hence $u_k$ is stable in $\mathcal{N}$. Then for any ball $B_r(p)$ with $r > 0$ and $p \in \Sigma_\infty$, there exists a constant $C > 0$ such that

$$\int_{\mathcal{N} \cap B_r(p)} \left[ \frac{1}{2R_k} |\nabla u_k|^2 + R_k W(u_k) \right] \leq C.$$

Because $u_k$ is stable in $\mathcal{N} \cap B_r(p)$, the stability of $\Sigma_\infty$ follows by applying the main result of [14].

The contradiction implies that $R_\alpha$ is bounded and the proposition is proven. □

Now $\{ u = 0 \} \setminus C_{R_2} = \cup_{\alpha} \Gamma_{\alpha}$, where each $\Gamma_{\alpha}$ is of Type I. Denote $\Gamma_{\alpha} \cap \{ r = R_2 \} = \{(R_2, z_{\alpha})\}$. After perhaps enlarging $R_2$, by Proposition 2.2, there is a positive lower bound for $|z_{\alpha} - z_{\beta}|$, $\forall \alpha \neq \beta$. Hence we can take the index $\alpha$ to be integers and we will relabel indices so that $z_\alpha < z_\beta$ for any $\alpha < \beta$. By continuity and the embeddedness of $\Gamma_{\alpha}$, it holds that $f_\alpha < f_\beta$ in $[R_2, +\infty)$ for any $\alpha < \beta$.

Define the functions

$$f_\alpha^+(r) := \frac{f_\alpha(r) + f_{\alpha+1}(r)}{2}, \quad \text{for } r \in [R_2, +\infty),$$

$$f_\alpha^-(r) := \frac{f_\alpha(r) + f_{\alpha-1}(r)}{2}, \quad \text{for } r \in [R_2, +\infty).$$

By definition, $f_\alpha^+ = f_{\alpha+1}^-$. In the above we take the convention that $f_\alpha^+(r) = +\infty$ (or $f_\alpha^-(r) = -\infty$) if there does not exist any other end lying above (respectively below) $\Gamma_{\alpha}$. Let

$$\mathcal{M}_\alpha := \{(r, z) : f_\alpha^-(r) < z < f_\alpha^+(r), \ r > R_2\}.$$

The following result describes the asymptotic behavior of $f_\alpha$ as $r \to +\infty$.

**Lemma 3.3.** There exists a constant $C$ such that for each $\alpha$, in $[R_2, +\infty)$ it holds that

$$\begin{cases}
|f_\alpha(r) - f_\alpha(R_2)| \leq C \log r, \\
|f_\alpha'(r)| \leq Cr^{-1}, \\
|f_\alpha''(r)| + |f_\alpha^{(3)}(r)| \leq Cr^{-2}.
\end{cases}$$

**Proof.** By Proposition 2.2 and implicit function theorem, for $t \in (-1 + b, 1 - b)$, there exists a continuous family of curves $\{ z = f_{\alpha,t}(r) \}$ satisfying $f_{\alpha,0} = f_\alpha$ and $u(r, f_{\alpha,t}(r)) \equiv t$ on $[R_2, +\infty)$. These curves will be denoted by $\Gamma_{\alpha,t}$. Next we divide the proof into four steps.

**Step 1.** Denote the second fundamental form of $\Gamma_{\alpha,t}$ by $A_{\alpha,t}$, the mean curvature by $H_{\alpha,t}$. By the bound on $A_{\alpha,t}$ in Theorem 2.4 and the axial symmetry of $\Gamma_{\alpha,t}$, there
exists a constant $C$ (independent of $t \in [-1 + b, 1 - b]$) such that for any $r \geq R_2$,

$$\left| f''_{\alpha,t}(r) \right| \leq \frac{C}{r}, \quad \left| f'_{\alpha,t}(r) \right| \leq C. \tag{3.3}$$

**Step 2.** For any $t \in [-1 + b, 1 - b]$ and $\lambda > 0$, let $\Sigma_{t,\lambda} := \lambda \Gamma_{\alpha,t} = \{ z = f_{\lambda,t}(r), r \geq \lambda R_2 \}$, where $f_{\lambda,t}(r) := \lambda f_{\alpha,t}(\lambda^{-1}r)$. By Theorem 2.4, as $\lambda \to 0$, $f_{\lambda,t}$ are uniformly bounded in $C_{loc}^{1,1}(0, +\infty)$. Hence after passing to a subsequence of $\lambda \to 0$, $f_{0,t}$ satisfies the minimal surface equation in the weak sense on $\mathbb{R}^n \setminus \{0\}$. It is directly verified that $f_{0,t} \equiv 0$. Since this is independent of the choice of subsequences of $\lambda \to 0$, we obtain

$$\lim_{r \to +\infty} f'_{\alpha,t}(r) = 0, \quad \forall t \in [-1 + b, 1 - b]. \tag{3.4}$$

**Step 3.** By the bound on mean curvature in Theorem 2.4, in $(R_2, +\infty)$, $f_{\alpha,t}$ satisfies

$$\frac{f''_{\alpha,t}(r)}{(1 + |f'_{\alpha,t}(r)|^2)^{3/2}} + \frac{n - 1}{r} \frac{f'_{\alpha,t}(r)}{(1 + |f'_{\alpha,t}(r)|^2)^{1/2}} = O(r^{-2}). \tag{3.5}$$

Combining this equation with (3.4), an ordinary differential equation analysis (e.g. by rewriting this ODE in the new variable $s := \log r)$ leads to the estimates for $f_{\alpha,t}$, $f'_{\alpha,t}$ and $f''_{\alpha,t}$, where the constant $C$ is independent of $t \in [-1 + b, 1 - b]$.

**Step 4.** Differentiating the relation $u(r, f_{\alpha,t}(r)) \equiv t$ in $r$ leads to

$$u_r + u_z f'_{\alpha,t} = 0. \tag{3.6}$$

Combining this relation with the estimate on $f'_{\alpha,t}$ gives

$$|u_r(r, z)| \leq C r^{-1}, \quad \text{in } \{|u| < 1 - b\} \cap \{r > R_2\}. \tag{3.7}$$

Differentiating (1.4) in $r$ we get the elliptic equation satisfied by $u_r$:

$$\Delta u_r = W''(u) u_r + \frac{n - 1}{r^2} u_r. \tag{3.8}$$

By standard interior gradient estimates we obtain

$$|u_{rz}(r, z)| \leq C r^{-1}, \quad \text{in } \{|u| < 1 - 2b\} \cap \{r > R_2\}. \tag{3.9}$$

Differentiating (3.6) in $r$ again, we obtain

$$u_{rr} + 2 u_{rz} f'_{\alpha,t} + u_{zz} |f'_{\alpha,t}|^2 + u_z f''_{\alpha,t} = 0. \tag{3.10}$$

Substituting (3.9) and the estimates on $f''_{\alpha,t}$ and $f'_{\alpha,t}$ into this identity, we obtain

$$|u_{rr}(r, z)| \leq C r^{-2}, \quad \text{in } \{|u| < 1 - 2b\} \cap \{r > R_2\}. \tag{3.10}$$

Differentiating (3.8) in $r$ once again, we obtain the equation for $u_{rr}$:

$$\Delta u_{rr} = W''(u) u_{rr} + W^{(3)}(u) u_r^2 + \frac{2(n - 1)}{r^2} u_{rr} + \frac{2(n - 1)}{r^3} u_r. \tag{3.11}$$

Applying standard interior gradient estimates we obtain

$$|u_{rrr}(r, z)| + \left| u_{rrz}(r, z) \right| \leq C r^{-2}, \quad \text{in } \{|u| < 1 - 3b\} \cap \{r > R_2\}. \tag{3.11}$$
Differentiating (3.6) twice in $r$ and evaluating at $t = 0$, we get
\[ u_{rrr} + 3u_{rrz}f'_α + 3u_{rzz}|f'_α|^2 + 2u_{rz}f''_α + u_{zzz}|f'_α|^3 + 3u_{zzz}f'_αf''_α + u_zf^{(3)}_α = 0. \]
Substituting (3.11) and estimates on $f'_α$, $f''_α$ into this equation, by noting that $|u_z|$ has a definite lower bound on $\Gamma_α \cap \{ r > R_2 \}$ (by combining Corollary 2.3 with (3.6)), we get the estimate on $f^{(3)}_α$.

Two corollaries follow from this lemma. First the bound on $f'_α$ implies an area growth bound for $\Gamma_α$.

**Corollary 3.4.** There exists a constant $C$ such that for each $\Gamma_α$, if $R$ is large enough,
\[ \text{Area}(\Gamma_α \cap (C_R \setminus C_{R_2})) \leq CR^n. \]

Next, by this lemma and Proposition 2.2, we obtain

**Corollary 3.5.** For each $\Gamma_α$, there exists an $\overline{R}_α$ such that $u_z$ has a definite sign in the open set $\{ (r, z) : r > \overline{R}_α, |z - f_α(r)| < 1 \}$.

**Proof.** By Proposition 2.2, there exists an $\overline{R}_{α,1}$ such that for any $r > \overline{R}_{α,1}$, $u$ is very close to a one dimensional solution $g(e_r \cdot (x, z) - \lambda)$ in $B_2((r, f_α(r)))$, where $e_r$ is a unit vector and $\lambda$ is a constant. In particular, there is only one connected component of $\{ u = 0 \}$ in this ball, which of course is just $\Gamma_α$. By Lemma 3.3, there exists another $\overline{R}_{α,2}$ such that if $r \geq \overline{R}_{α,2}$, then $\{ u = 0 \} \cap B_2((r, f_α(r)))$ is very close to the line $\{ z = f_α(r) \}$. Combining these two facts, we see if $r > \overline{R}_α := \max\{ \overline{R}_{α,1}, \overline{R}_{α,2} \}$, then $e_r$ is very close to the $z$ direction or the minus $z$ direction. Because $g'$ has a definite positive lower bound in any compact set of $\mathbb{R}$, $u_z$ has a definite sign in $B_1((r, f_α(r)))$. Since $\{ (r, z) : r > \overline{R}_α, |z - f_α(r)| < 1 \}$ is a connected open set, the conclusion follows.

The following lemma gives a growth bound of the energy localized in the domain around each end.

**Lemma 3.6.** For any $α$, there exists a constant $C_α$ such that
\[ \int_{\mathcal{M}_α \cap (B_R^{α+1} \setminus C_{R_2})} \left[ \frac{1}{2} |\nabla u|^2 + W(u) \right] \leq C_α R^n, \quad \forall R > R_2. \]

**Proof.** This growth bound follows from the following two estimates.

**Claim 1.** For any $L > 0$ and $R > 0$,
\[ \int_{\{ R_2 < r < R, f_α(r) - L < z < f_α(r) + L \}} \left[ \frac{1}{2} |\nabla u|^2 + W(u) \right] \leq C_α LR^n. \]
This follows by combining the trivial bound $\frac{1}{2} |\nabla u|^2 + W(u) \leq C$, co-area formula and the area growth bound in Corollary 3.4.

**Claim 2.** If $L$ is sufficiently large,
\[ \int_{\{ R_2 < r < R, f_α(r) + L < z < f_{α+1}(r) - L \}} \left[ \frac{1}{2} |\nabla u|^2 + W(u) \right] \leq C_α R^n. \]
Note that here $f_{α+1}$ could be $+\infty$ everywhere, that is, there is no end lying above $\Gamma_α$. 
Assume the constant $b > 0$ has been chosen so small that $W''(u) \geq c$ in $\{|u| > 1-2b\}$ for a positive constant $c$. A direct calculation leads to the following differential inequality in $\{|u| > 1-2b\}$:

$$\Delta \left[ \frac{1}{2} |\nabla u|^2 + W(u) \right] \geq c \left[ \frac{1}{2} |\nabla u|^2 + W(u) \right].$$  \hspace{1cm} (3.12)

Note that if we have chosen $L$ large enough, then $\{R_2 < r < R, f_\alpha(r) + L/2 < z < f_{\alpha+1}(r) - L/2\} \subset \{|u| > 1-b\}$. Without loss of generality, assume $u > 1-b$ in $\{R_2 < r < R, f_\alpha(r) + L/2 < z < f_{\alpha+1}(r) - L/2\}$.

Take a smooth function $\zeta \in C^\infty(\mathbb{R})$ satisfying $0 \leq \zeta \leq 1$, $\zeta \equiv 1$ in $(1-b, +\infty)$, $\zeta \equiv 0$ in $(-\infty, 1-2b)$ and $|\zeta'|^2 + |\zeta''|^2 \leq 100b^{-2}$.

Multiplying (3.12) by $\zeta(u)$ and integrating in the domain $\mathcal{D} := \{f_\alpha(r) < z < f_{\alpha+1}(r)\} \cap B^{n+1}_R(0) \setminus \mathcal{C}_{R_2}$, we obtain

$$\int_{\{R_2 < r < R, f_\alpha(r) + L/2 < z < f_{\alpha+1}(r) - L/2\} \cap B^{n+1}_R} \left[ \frac{1}{2} |\nabla u|^2 + W(u) \right]$$

$$\leq \int_{\mathcal{D}} \left[ \frac{1}{2} |\nabla u|^2 + W(u) \right] \zeta(u) \quad \text{(by our choice of } \zeta)$$

$$\leq C \int_{\mathcal{D}} \Delta \left[ \frac{1}{2} |\nabla u|^2 + W(u) \right] \zeta(u) \quad \text{(by (3.12))}$$

$$= C \int_{\mathcal{D}} \left[ \frac{1}{2} |\nabla u|^2 + W(u) \right] \left[ \zeta'(u) \Delta u + \zeta''(u) |\nabla u|^2 \right] \quad \text{(integration by parts)}$$

+ boundary integrals on $\partial \mathcal{C}_{R_2} \cap \mathcal{D}$

+ boundary integrals on $\partial B^{n+1}_R \cap \mathcal{D}$

$$\leq C \int_{\mathcal{D}} \left[ \frac{1}{2} |\nabla u|^2 + W(u) \right] \left( |\zeta'(u)| + |\zeta''(u)| \right)$$

+ boundary integrals on $\partial \mathcal{C}_{R_2} \cap \mathcal{D}$

+ boundary integrals on $\partial B^{n+1}_R \cap \mathcal{D}$.

In the right hand of this inequality, the following estimates hold.

- Because $\zeta'(u)$ and $\zeta''(u)$ are nonzero only in $\{R_2 < r < R, f_\alpha(r) < z < f_{\alpha+1}(r) - L\}$ and $\{R_2 < r < R, f_\alpha(r) + L < z < f_{\alpha+1}(r)\}$ (if this set is non-empty), estimate on the first integral follows by applying Claim 1.

- The boundary integral on $\partial \mathcal{C}_{R_2} \cap \mathcal{D}$ is of the order $O(R)$, because the area of $\partial \mathcal{C}_{R_2} \cap \mathcal{D}$ is of the order $O(R)$ and the integrands are of the order $O(1)$.

- Finally, the boundary integral on $\partial B^{n+1}_R \cap \mathcal{D}$ is of the order $O(R^n)$, because the area of $\partial B^{n+1}_R \cap \mathcal{D}$ is of the order $O(R^n)$ and the integrands are of the order $O(1)$.

This completes the proof of Claim 2.  \hspace{1cm} \Box
4. A Toda system

In this section, keeping the notations used in the previous section, \( u \) denotes an axially symmetric solution of (1.1) in \( \mathbb{R}^{n+1} \) satisfying for some \( R_2 > 0 \), it is stable outside the cylinder \( C_{R_2} \) and

\[
\{ u = 0 \} \setminus C_{R_2} = \cup_{\alpha \in \mathbb{Z}} \Gamma_\alpha, \quad \Gamma_\alpha := \{ z = f_\alpha(r), r > R_2 \},
\]

where \( f_\alpha \in C^4([R_2, +\infty)) \) and they are increasing in \( \alpha \in \mathbb{Z} \).

4.1. Fermi coordinates. In this subsection we introduce Fermi coordinates with respect to \( \Gamma_\alpha \).

Since \( \Gamma_\alpha \) is the graph of \( f_\alpha \), we will use \( \ell \rightarrow (\ell, f_\alpha(\ell)) \) as a parametrization of \( \Gamma_\alpha \). The upward unit normal vector of \( \Gamma_\alpha \) at \( (\ell, f_\alpha(\ell)) \) is

\[
N_\alpha(\ell) := \frac{1}{\sqrt{1 + |f'_\alpha(\ell)|^2}} (-f'_\alpha(\ell) \partial_r + \partial_z).
\]

The second fundamental form of \( \Gamma_\alpha \) at \( (\ell, f_\alpha(\ell)) \) with respect to \( N_\alpha(\ell) \) is denoted by \( A_\alpha(\ell) \). The principal curvatures are given by

\[
\begin{align*}
\kappa_{\alpha,i}(\ell) &= \frac{-1}{\ell} \frac{f''_\alpha(\ell)}{\sqrt{1 + |f'_\alpha(\ell)|^2}}, & 1 \leq i \leq n - 1, \\
\kappa_{\alpha,n}(\ell) &= \frac{-1}{\ell} \frac{f''_\alpha(\ell)}{\sqrt{1 + |f'_\alpha(\ell)|^2}}.
\end{align*}
\]

By Lemma 3.3, we have

\[
|A_\alpha(\ell)| + |A'_\alpha(\ell)| \leq C\ell^{-3/2}, \quad \forall \ell \geq R_2.
\]

Let \((\ell, t)\) be the Fermi coordinates with respect to \( \Gamma_\alpha \), that is, for any point \( X \) lying in a neighborhood of \( \Gamma_\alpha \), take \((\ell, f_\alpha(\ell)) \in \Gamma_\alpha \) to be the nearest point to \( X \) and \( t \) be the signed distance of \( X \) to \( \Gamma_\alpha \) (positive above \( \Gamma_\alpha \)). By Theorem 2.4, these are well defined in the open set \( \{(\ell, t) : |t| < c_F \ell, \ell > R_2\} \) for a constant \( c_F > 0 \).

For each \( t \), let \( \Gamma_{\alpha,t}^t \) be the smooth hypersurface where the signed distance to \( \Gamma_\alpha \) equals \( t \). The mean curvature of \( \Gamma_{\alpha,t}^t \) has the form

\[
H_{\alpha}(\ell,t) = \sum_{i=1}^{n} \frac{\kappa_{\alpha,i}(\ell)}{1 - t\kappa_{\alpha,i}(\ell)} = H_{\alpha}(\ell) + O\left(|t||A_\alpha(\ell)||^{2}\right) = H_{\alpha}(\ell) + O\left(|t|\ell^{-3}\right),
\]

where in the last step we have used (4.2).

Denote by \( \Delta_{\alpha,t} \) the Beltrami-Laplace operator with respect to the induced metric on \( \Gamma_{\alpha,t}^t \). In Fermi coordinates the Euclidean Laplace operator has the form

\[
\Delta = \Delta_{\alpha,t} - H_{\alpha}(\ell,t)\partial_t + \partial_{tt}.
\]

Concerning the error between \( \Delta_{\alpha,t} \) and \( \Delta_{\alpha,0} \), we have (see for example [17, Lemma 3.3])

**Lemma 4.1.** Suppose \( \varphi \) is a \( C^2 \) function of \( \ell \) only, then

\[
|\Delta_{\alpha,t}\varphi(\ell) - \Delta_{\alpha,0}\varphi(\ell)| \leq C\ell^{-3/2}|t| (||\varphi''(\ell)|| + |\varphi'(\ell)|), \quad \forall t \in (-c_F \ell, c_F \ell).
\]
Note that here, in order to get \( \ell^{-3/2} \) in the right hand side of (4.5), we have used Lemma 3.3 and the estimate (4.2) again.

We introduce some notations.

- For \( \ell > R \), let \( D^\pm_\alpha(\ell) \) be the distance of \( (\ell, f_\alpha(\ell)) \) to \( \Gamma_{\alpha \pm 1} \), respectively.
- Denote \( D_\alpha(\ell) := \min \{ D^+_\alpha(\ell), D^-_\alpha(\ell) \} \).
- \( M(\ell) := \max_{\alpha} \max_{s \geq \ell} e^{-\sqrt{2}D_\alpha(s)} \).

By Lemma 3.3, \( \Gamma_\alpha \) and \( \Gamma_{\alpha + 1} \) are almost parallel. Proceeding as in the proof of [18, Lemma 8.3] we get

**Lemma 4.2.** For any \( \ell > R_2 \),

\[
\begin{align*}
D^+_\alpha(\ell) &= f_{\alpha + 1}(\ell) - f_\alpha(\ell) + O(\ell^{-1/6}), \\
D^-_\alpha(\ell) &= f_\alpha(\ell) - f_{\alpha - 1}(\ell) + O(\ell^{-1/6}).
\end{align*}
\]

### 4.2 Optimal approximation

Fix a function \( \zeta \in C^\infty_0(-2,2) \) with \( \zeta \equiv 1 \) in \((-1,1)\), \( |\zeta'| + |\zeta''| \leq 16 \). For all \( \ell \) large, let (to ease notation, dependence on \( \ell \) will not be written down)

\[
\bar{g}(t) = \zeta(8(\log \ell)t)g(t) + [1 - \zeta(8(\log \ell)t)]\text{sgn}(t), \quad t \in (-\infty, +\infty).
\]

In particular, \( \bar{g} \equiv 1 \) in \((16\log \ell, +\infty)\) and \( \bar{g} \equiv -1 \) in \((-\infty, -16\log \ell)\).

\( \bar{g} \) is an approximate solution to the one dimensional Allen-Cahn equation, that is,

\[
g''(t) = W'(\bar{g}(t)) + \bar{\xi}(t),
\]

where \( \text{spt}(\bar{\xi}) \in \{8\log \ell < |t| < 16\log \ell\} \), and \( |\bar{\xi}| + |\bar{\xi}'| + |\bar{\xi}''| \lesssim \ell^{-4} \). Hereafter we use the notation \( A \lesssim B \) for \( A \leq CB \) if \( C \) is a universal constant.

In the following we assume \( u \) has the same sign as \((-1)^\alpha\) between \( \Gamma_\alpha \) and \( \Gamma_{\alpha + 1} \).

**Lemma 4.3.** For any \( \ell > R_2 \) (perhaps after enlarging \( R_2 \)) and \( \alpha \in \mathbb{Z} \), there exists a unique \( h_\alpha(\ell) \) such that in the Fermi coordinates with respect to \( \Gamma_\alpha \),

\[
\int_{-\infty}^{+\infty} [u(\ell, t) - g_*(\ell, t)] g'(t - h_\alpha(\ell)) \, dt = 0,
\]

where for each \( \alpha \), in \( M_\alpha \) we define

\[
g_*(\ell, t) := g_\alpha + \sum_{\beta < \alpha} [g_\beta - (-1)^\beta] + \sum_{\beta > \alpha} [g_\beta + (-1)^\beta],
\]

and in the Fermi coordinates \( (\ell, t) \) with respect to \( \Gamma_\beta \),

\[
g_\beta(\ell, t) := \bar{g}((-1)^\beta (t - h_\beta(\ell))).
\]

Moreover, for any \( \alpha \in \mathbb{Z} \),

\[
\lim_{\ell \to +\infty} (|h_\alpha(\ell)| + |h'_\alpha(\ell)| + |h''_\alpha(\ell)| + |h^{(3)}_\alpha(\ell)|) = 0.
\]

The proof of this lemma is similar to the one for [17, Proposition 4.1], although now there may be infinitely many components. Indeed, we can define a nonlinear map on \( \bigoplus_\alpha C(\Gamma_\alpha) \) as

\[
F(h) := \left( \int_{-\infty}^{+\infty} [u(\ell, t) - g_*(\ell, t; h)] g'_\alpha(t, h) \, dt \right).
\]
The $\alpha$ component of its derivative depends only on finitely many $\beta$, i.e. $DF(h)$ has finite width with respect to the index set. Moreover, it is diagonally dominated and hence invertible. This lemma then follows from the inverse function theorem.

Let $g_\alpha$ and $g_*$ be as in this lemma. Define $\phi := u - g_*$. In Fermi coordinates with respect to $\Gamma_\alpha$, the equation for $\phi$ reads as

$$
\Delta_\alpha, t \phi - H_\alpha \frac{\partial}{\partial t} \phi + \partial_t \phi
= W''(g_\ast) \phi + \mathcal{N}(\phi) + \mathcal{I} + \sum_{\beta \neq \alpha} \left( (-1)^\beta g'_\alpha \mathcal{R}_{\beta,1} - g''_\alpha \mathcal{R}_{\beta,2} \right) - \sum_\beta \xi_\beta,
$$

(4.7)

where

$$
\mathcal{N}(\phi) = W''(g_\ast + \phi) - W''(g_\ast) - W''(g_\ast) \phi = O(\phi^2),
$$

$$
\mathcal{I} = W'(g_\ast) - \sum_\beta W'(g_\beta),
$$

while for each $\beta$, in the Fermi coordinates with respect to $\Gamma_\beta$,

$$
\xi_\beta(\ell, t) = \bar{\xi} \left( (-1)^\beta (t - h_\beta(\ell)) \right),
$$

$$
\mathcal{R}_{\beta,1}(\ell, t) := H_\beta(\ell, t) + \Delta_\beta, t h_\beta(\ell),
$$

$$
\mathcal{R}_{\beta,2}(\ell, t) := |\nabla_\beta, t h_\beta(\ell)|^2.
$$

As in [17, Lemma 4.6], because $u = 0$ on $\Gamma_\alpha$, $h_\alpha$ can be controlled by $\phi$ in the following way.

**Lemma 4.4.** For each $\alpha$ and $r > R_2$, we have

$$
\|h_\alpha\|_{C^{2,1/2}(\ell, +\infty)} \lesssim \|\phi\|_{C^{2,1/2}(c_{\ell}^r)} + M(\ell),
$$

(4.8)

$$
\|h'_\alpha\|_{C^{1,1/2}(\ell, +\infty)} \lesssim \|\phi_r\|_{C^{1,1/2}(c_{\ell}^r)} + \ell^{-1/6} M(\ell).
$$

(4.9)

**4.3. Toda system.** By [17, Section 5], we get the following Toda system

$$
H_\alpha + \Delta_\alpha, 0 h_\alpha = \frac{2A^2}{\sigma_0} \left( e^{-\sqrt{2}D_\alpha} - e^{-\sqrt{2}D_\alpha} \right) + E_\alpha,
$$

(4.10)

where $E_\alpha$ is a higher order error term. More precisely, [17, Lemma 5.1] reads in our specific setting as

**Lemma 4.5.** For any $\ell > 2R_2$,

$$
|E_\alpha(\ell)| \lesssim \ell^{-3} + \ell^{-1/2} M(\ell - 100 \log \ell) + M(\ell - 100 \log \ell)^2
$$

(4.11)

$$
+ \max_\beta \|H_\beta + \Delta_\beta, 0 h_\beta\|_{C^{1/2}(\ell - 100 \log \ell, +\infty)}^2 + \|\phi\|_{C^{2,1/2}(\ell - 100 \log \ell, +\infty)}^2.
$$

Here it is still useful to note that by (4.2), now we can take the upper bound on the second fundamental form to be $O\left(\ell^{-3/2}\right)$ when using the derivation in [17].
4.4. Estimates on $\phi$. Arguing exactly in the same way as in [17, Section 6], we have

**Lemma 4.6.** There exist two constants $C$ such that for all $\ell$ large,

$$
\max_{\alpha} \|H_{\alpha} + \Delta_{\alpha,0} h_{\alpha}\|_{C^{1/2}(\ell, +\infty)} + \|\phi\|_{C^{2,1/2}(C_{\ell}^c)} \\
\leq \frac{1}{2} \left[ \max_{\alpha} \|H_{\alpha} + \Delta_{\alpha,0} h_{\alpha}\|_{C^{1/2}(\ell - 100 \log \ell, +\infty)} + \|\phi\|_{C^{2,1/2}(C_{\ell - 100 \log \ell}^c)} \right] + CM (\ell - 100 \log \ell) + C\ell^{-3}.
$$

The constant $1/2$ in the right hand side of this inequality allows us to repeat the iteration argument used in the proof of [9, Lemma 11]. This results in the estimate

$$
|H_{\alpha}(\ell) + \Delta_{\alpha,0} h_{\alpha}(\ell)| + \|\phi\|_{C^{2,1/2}(C_{\ell}^c)} \leq C \left[ \ell^{-3} + M (\ell - 100 \log \ell) \right], \quad \forall \ell \geq R_2.
$$

By [17, Proposition 10.1]), $M(\ell) \lesssim \ell^{-2}$. Hence

(4.12) \quad $|H_{\alpha}(\ell) + \Delta_{\alpha,0} h_{\alpha}(\ell)| + \|\phi\|_{C^{2,1/2}(C_{\ell}^c)} \leq C\ell^{-2}.$

Next by [17, Proposition 7.1], we have an improved estimate on the horizontal derivative

(4.13) \quad $\|\phi_{\ell}\|_{C^{1,1/2}(C_{\ell}^c)} \leq C\ell^{-2-1/7}.$

In view of Lemma 4.4, (4.13) gives

(4.14) \quad $\|h_{\alpha}'\|_{C^{1,1/2}(\ell, +\infty)} \leq C\ell^{-2-1/7}.$

Substituting this into (4.10) and applying Lemma 3.3, we obtain

(4.15) \quad \int_{R_2}^{+\infty} e^{-\sqrt{2}(f_0(r) - f_{a-1}(r))} \eta(r) \frac{A^2}{\sigma_0} \right) + O \left( r^{-2-1/7} \right).

Finally, the reduced stability condition (see [17, Proposition 8.1]) now reads as

**Proposition 4.7.** For any $\eta \in C_0^\infty (R_2, +\infty)$, we have

(4.16) \quad \frac{4\sqrt{2} A^2}{\sigma_0} \int_{R_2}^{+\infty} e^{-\sqrt{2}(f_0(r) - f_{a-1}(r))} \eta(r) r^{n-1} dr \leq \left[ 1 + CR_2^{-1/2} \right] \int_{R_2}^{+\infty} |\eta'(r)| r^{n-1} dr + C \int_{R_2}^{+\infty} \eta(r) r^{n-2-1/2} dr.

5. The Case $3 \leq n \leq 9$: Proof of Theorem 1.3

In this section we keep the same setting as in the previous section, with the additional assumption that $3 \leq n \leq 9$. In order to prove Theorem 1.3, we argue by contradiction and assume there are at least two ends of $u$. We show this assumption leads to a contradiction if $3 \leq n \leq 9$.

Take two adjacent ends $\Gamma_{a-1}$ and $\Gamma_a$. Let $v_a := f_a - f_{a-1}$ and $V_a := e^{-\sqrt{2} v_a}$. By (4.15) we get a constant $\mu \in (0, 1/8)$ such that

(5.1) \quad v_a''(r) + \frac{n-1}{r} v_a'(r) \leq \frac{4A^2}{\sigma_0} e^{-\sqrt{2} v_a} + O \left( r^{-2-\mu} \right), \quad \text{in} \ (R_2, +\infty).
Consequently,

\begin{equation}
(5.2) \quad - V''_\alpha - \frac{n-1}{r} V'_\alpha \leq \frac{4\sqrt{2}A^2}{\sigma_0} V_\alpha^2 - V^{-1}_\alpha |V'|^2 + O(r^{-2-\mu}) V_\alpha, \quad \text{in } (R_2, +\infty).
\end{equation}

For any \( q > 0 \) and \( \eta \in C_c^\infty(0, \infty) \), multiplying (5.2) by \( V_\alpha(r)^{2q-1} \eta(r)^2 r^{n-1} \) and integrating by parts leads to

\begin{equation}
(5.3) \quad \leq \frac{4\sqrt{2}A^2}{\sigma_0} \int_{R_2}^{+\infty} V_\alpha(r)^{2q-2} |V'_\alpha(r)|^2 \eta(r)^2 r^{n-1} dr
\end{equation}

\[ + C \int_{R_2}^{+\infty} V_\alpha(r)^{2q} \left[ |\eta'(r)|^2 + \eta(r) |\eta''(r)| + \eta(r)^2 r^{-2-\mu} \right] r^{n-1} dr. \]

On the other hand, substituting \( V_\alpha^q \eta \) as test function into (4.16) leads to

\begin{equation}
(5.4) \quad \leq q^2 \int_{R_2}^{+\infty} V_\alpha(r)^{2q-1} \eta(r)^2 r^{n-1} dr
\end{equation}

\[ + C \int_{R_2}^{+\infty} V_\alpha(r)^{2q} \left[ |\eta'(r)|^2 + \eta(r) |\eta''(r)| + \eta(r)^2 r^{-2-\mu} \right] r^{n-1} dr. \]

Combining (5.3) and (5.4), if \( q < 2 \) and \( R_2 \) is sufficiently large, we get a constant \( C(q) \) such that

\begin{equation}
(5.5) \quad \int_{R_2}^{+\infty} V_\alpha(r)^{2q+1} \eta(r)^2 r^{n-1} dr
\end{equation}

\[ \leq C(q) \int_{R_2}^{+\infty} V_\alpha(r)^{2q} \left[ |\eta'(r)|^2 + \eta(r) |\eta''(r)| + \eta(r)^2 r^{-2-\mu} \right] r^{n-1} dr. \]

If \( 0 \leq \eta \leq 1 \), following Farina [?], replacing \( \eta \) by \( \eta^m \) for some \( m \gg 1 \) and then applying Hölder inequality to (5.5) we get

\begin{equation}
(5.6) \quad \int_{R_2}^{+\infty} V_\alpha(r)^{2q+1} \eta(r)^{2m} r^{n-1} dr \leq C(q) \int_{R_2}^{+\infty} \left[ |\eta'(r)|^2 + |\eta''(r)| + r^{-2-\mu} \right]^{2q+1} r^{n-1} dr.
\end{equation}

For any \( R > 2R_2 \), take \( \eta_R \in C_c^\infty(R_2, 2R) \) such that \( 0 \leq \eta_R \leq 1, \eta_R \equiv 1 \) in \((2R_2, R)\), \( |\eta'_R|^2 + |\eta''_R|^2 \leq 16 R^{-2} \) in \((R, 2R)\). Substituting \( \eta_R \) into (5.6), we get

\begin{equation}
(5.7) \quad \int_{2R_2}^{R} V_\alpha(r)^{2q+1} r^{n-1} dr \leq C + CR^{n-2(2q+1)}.
\end{equation}

Since \( n \leq 9 \), we can take \( 2q + 1 = n/2 \). After letting \( R \to +\infty \) in (5.7) we arrive at

\begin{equation}
(5.8) \quad \int_{2R_2}^{+\infty} V_\alpha(r)^{n/2} r^{n-1} dr \leq C.
\end{equation}
As in Dancer-Farina [4], this implies that
\[
\lim_{r \to +\infty} r^2 e^{-\sqrt{2}v_\alpha(r)} = 0,
\]
which then leads to a contradiction by applying (5.1) exactly in the same way as in [4] (see also [16] for the corresponding result for Toda system), if \( n \geq 3 \).

In other words, there is only one end of \( u \). The one dimensional symmetry of \( u \) follows, for example by applying the main results of [10] and [15], because now we have the energy growth bound from Lemma 3.6.

6. The case \( n = 2 \): Proof of Theorem 1.4

In this section \( u \) denotes an axially symmetric solution of (1.1) in \( \mathbb{R}^3 \), which is stable outside \( B_{R_*}^2(0) \times (-R_*, R_*) \). Hence there exists a positive function \( \varphi \in C^2(\mathbb{R}^3) \) such that
\[
(6.1) \quad \Delta \varphi = W''(u) \varphi
\]
outside \( B_{R_*}^2(0) \times (-R_*, R_*) \).

By a direct differentiation we see \( u_z \) satisfies the linearized equation (6.1). We will show

**Lemma 6.1.** Any nodal domain of \( u_z \) is not disjoint from \( B_{R_*}^2(0) \times (-R_*, R_*) \).

Before proving this lemma, let us first present some technical results.

Keeping notations as in Section 3 and Section 4, we define for each \( \alpha \),
\[
\mathcal{N}_\alpha := \left\{ X : -\frac{3}{4}D^-_\alpha(\Pi_\alpha(X)) < d_\alpha(X) < \frac{3}{4}D^+_\alpha(\Pi_\alpha(X)) \right\},
\]
where \( \Pi_\alpha(X) \) is the nearest point to \( X \) on \( \Gamma_\alpha \) and \( d_\alpha \) is the signed distance to \( \Gamma_\alpha \).

By Theorem 2.4 and Lemma 3.3, \( \Pi_\alpha \) is well defined and smooth in the open set \( \{ (r,z) : |d_\alpha(r,z)| < c_Fr, r > R_* \} \) after perhaps enlarging \( R_* \).

**Lemma 6.2.** For each \( \alpha \), there exists an \( R^*_\alpha > R_* \) so that the following holds.

(i) There is a connected component \( \Omega_\alpha \) of \( \{ u_z \neq 0 \} \cap \{ r > R^*_\alpha \} \), which contains \( \Gamma_\alpha \cap \{ r > R^*_\alpha \} \) and is contained in \( \mathcal{N}_\alpha \).

(ii) There exists a constant \( C_\alpha \) such that
\[
(6.2) \quad \int_{\Omega_\alpha \cap \mathcal{C}_R} u_z^2 \leq C_\alpha R^2, \quad \forall R > R^*_\alpha.
\]

**Proof.** (i) This follows by looking at the distance type function. Indeed, for any \( (r_*, z_*) \in \Gamma_\alpha \) where \( r_* \) is large, let \( \varepsilon := \max\{D^-_\alpha(r_*^{-1}), r_*^{-1}\} \) and
\[
u_\varepsilon(r,z) := u(r_* + \varepsilon^{-1}r, z_\varepsilon + \varepsilon^{-1}z).
\]

By Proposition 2.2,
\[
(6.3) \quad \lim_{r \to +\infty} D^\pm_\alpha(r) = +\infty.
\]

Hence \( \varepsilon \ll 1 \) if \( r_* \gg 1 \).
As in [7], consider the signed distance type function $\Psi_\varepsilon$, which is defined by the relation

$$u_\varepsilon = g \left( \frac{\Psi_\varepsilon}{\varepsilon} \right).$$

By the vanishing viscosity method (see for example [15, Appendix A]) and the convergence of $\{u_\varepsilon = 0\}$ (by Lemma 3.3), as $\varepsilon \to 0$, in any compact set of $\{-1 \leq r \leq 1, -1 \leq z \leq 1\}$, $\Psi_\varepsilon$ converges uniformly to

$$\Psi_\infty(r, z) := \begin{cases} 
1 - z, & 1/2 \leq z \leq 1, \\
1/2 - z, & -1/2 \leq z \leq 1/2, \\
-1 - z, & -1 \leq z \leq -1/2.
\end{cases}$$

Moreover, because $\Psi_\infty$ is $C^1$ in $\{-1 < r < 1, -1/2 < z < 1/2\}$, $\Psi_\varepsilon$ converges in $C^1(\{|r| < 1/2, -1/4 < z < 1/4\})$. In particular, for all $\varepsilon$ small,

$$\frac{\partial u_\varepsilon}{\partial z} = \frac{1}{\varepsilon} g' \left( \frac{\Psi_\varepsilon}{\varepsilon} \right) \frac{\partial \Psi_\varepsilon}{\partial z} < 0, \quad \text{in } \{|r| < 1/2, -1/4 < z < 1/4\}.$$

Similarly, $\frac{\partial u_\varepsilon}{\partial z} > 0$ in $\{|r| < 1/2, -4/5 < z < -3/4\} \cup \{|r| < 1/2, 3/4 < z < 4/5\}$. Rescaling back we get the conclusion.

(ii) This follows by adding the estimates of Lemma 3.6 in $\alpha, \alpha + 1$ and $\alpha - 1$. □

**Lemma 6.3.** Suppose $\Omega$ is a nodal domain of $u_z$, which is disjoint from $B^2_{R_\ast}(0) \times (-R_\ast,R_\ast)$. Then

$$\limsup_{r \to +\infty} \frac{1}{r^2} \int_{\Omega \cap B_r(0)} u_z^2 = +\infty.$$

**Proof.** Assume by the contrary, there exists a constant $C$ such that for all $r$ large,

$$\int_{\Omega \cap B_r(0)} u_z^2 \leq Cr^2.$$

Then the standard Liouville type theorem applies to the degenerate equation (see [8, 2])

$$\text{div} \left( \varphi^2 \nabla \frac{u_z}{\varphi} \right) = 0,$$

which implies that $u_z \equiv 0$ in $\Omega$. This is a contradiction. □

**Proof of Lemma 6.1.** Assume by the contrary, there is a nodal domain of $u_z$ disjoint from $B^2_{R_\ast}(0) \times (-R_\ast,R_\ast)$. Denote it by $\Omega$ and assume without loss of generality $u_z > 0$ in $\Omega$. Since for any $R, r > 0$,

$$|C_R \cap B_r(0)| \leq CR^2 r,$$

Lemma 6.3 implies that $\Omega$ cannot be totally contained in $C_R$. In other words, $\Omega$ is unbounded in the $r$ direction.

Let $\Omega_\alpha$ be defined as in Lemma 6.2. Then we claim that

**Claim.** There exists at most one $\alpha$ such that $\Omega_\alpha \subset \Omega$.

To prove this claim, we assume by the contrary that there are $\alpha \neq \beta$ such that
\(\Omega_\alpha \cup \Omega_\beta \subset \Omega\). Since \(u_z > 0\) in \(\Omega_\alpha \cup \Omega_\beta\), \(|\alpha - \beta| \geq 2\). In particular, there exists a \(\gamma\) lying between \(\alpha\) and \(\beta\), and \(u_z < 0\) in \(\Omega_\gamma\).

Let \(\tilde{\Omega}\) be the nodal domain of \(u_z\) containing \(\Omega_\gamma\). Viewing all of these domains as open sets in the \((r,z)\) plane, \(\Omega_\alpha\) and \(\Omega_\beta\) can be connected by a continuous curve totally contained in \(\Omega\), which together with \(\Gamma_\alpha\) and \(\Gamma_\beta\) forms a simple unbounded Jordan curve. This curve divides the plane into at least two domains, \(\tilde{\Omega}\) lying on one side and \(B_{R_*}^2(0) \times (-R, R)\) on the other side.

Then there are only finite many of ends of \(u\) in \(\tilde{\Omega}\), and we can add the estimates in Lemma 3.6 to arrive at

\[
\int_{\tilde{\Omega} \cap B_R(0)} |\nabla u|^2 \leq C_{\alpha\beta} R^2, \quad \forall R \text{ large.}
\]

This is a contradiction with Lemma 6.3, which finishes the proof of the Claim.

By this Claim, there exists an \(R_3 > 0\) such that \(\Omega \cap \{r > R_3\} \subset \{f_{\alpha - 1}(r) < z < f_{\alpha + 1}(r)\}\). Using Lemma 3.6 again, we get a constant \(C\) such that

\[
\int_{\Omega \cap B_R(0)} |\nabla u|^2 \leq C R^2, \quad \forall R \text{ large.}
\]

Since \(\Omega\) is assumed to be disjoint from \(B_{R_*}^2(0) \times (-R, R)\), applying Lemma 6.3 again we get a contradiction. This completes the proof. \(\square\)

Since \(u\) is smooth, the number of connected components of \(\{u_z \neq 0\} \cap B_{2R_*}(0)\) is finite. Then by the above lemma we obtain

**Corollary 6.4.** There are only finitely many nodal domains of \(u_z\).

Now we come to the proof of Theorem 1.4.

**Proof of Theorem 1.4.** By the previous corollary, nodal domains of \(u_z\) are denoted by \(\Omega^m\), \(m = 1, \ldots, N\) for some \(N \in \mathbb{N}\).

Assume there are infinitely many ends, \(\Gamma_\alpha\). These ends are divided into \(N\) classes, \(I_m (1 \leq m \leq N)\), that is, \(\Gamma_\alpha \in I_m\) if \(\Omega_\alpha \subset \Omega^m\).

There is a class, say \(I_1\), containing infinitely many ends. Take two indices \(\alpha, \beta \in I_1\) which are adjacent in \(I_1\). \(\Gamma_\alpha\) and \(\Gamma_\beta\) are connected by a curve in \(\Omega^1\), together with \(\Gamma_\alpha\) and \(\Gamma_\beta\) which gives a simple unbounded Jordan curve \(\gamma_{\alpha\beta}\) in the plane. This curve divides the \((r, z)\) plane into at least two open domains. Since \(u_z\) has the same sign in \(\Omega_\alpha\) and \(\Omega_\beta\), there exists a \(\Gamma_\gamma\) lying between \(\Gamma_\alpha\) and \(\Gamma_\beta\). Assume \(\Omega_\gamma \subset \Omega^{M(\alpha)}\). This defines a map from \(I_1\) to \(\{1, \ldots, N\}\). Moreover, if \(\alpha, \beta \in I_1\) and \(\alpha \neq \beta\), then \(M(\alpha) \neq M(\beta)\), in other words, \(\Omega^{M(\alpha)}\) and \(\Omega^{M(\beta)}\) lie on two sides of a simple Jordan curve totally contained in \(\Omega^1\). This leads to a contradiction because \(I_1\) is an infinite set.

Once we know that there are only finitely many ends, by Lemma 3.6 we obtain a constant \(C\) such that

\[
\int_{B_R(0) \setminus C_{R_*}} \left[ \frac{1}{2} |\nabla u|^2 + W(u) \right] \leq CR^2, \quad \forall R > R_*.
\]
On the other hand,
\[
\int_{B_R(0) \cap C_{R_\ast}} \left[ \frac{1}{2} |\nabla u|^2 + W(u) \right] \leq C |B_R(0) \cap C_{R_\ast}| \leq CR_\ast^2 R, \quad \forall R > R_\ast.
\]
Combining these two estimates we get (1.3).

Finally, since there are only finitely many ends, by Lemma 3.3, there exist two constants \(C_4, R_4 > 0\) such that \(\{u = 0\} \setminus C_{R_4} \subset \{|z| < C_4 r\}\). From this we see the existence of \(R > 0\) such that \(u\) does not change sign in \(C_R \cap \{|z| > R\}\). □

7. Bound on number of ends: Proof of Theorems 1.5 and 1.6

In this section, by using the nodal domain information of direction derivatives (translation Jacobi field), we deduce a relation between Morse index and the number of ends. We mainly rely on information about \(u_z\) (just as in the previous section), which is almost along the normal direction of each end (by Lemma 3.3). For the proof of Theorem 1.6, another condition on the sign of \(u_r\) is needed. This sign condition will follow by combining the nodal information of \(u_z\) or \(u_y\) and the fact that \(u_x = \frac{2r}{r^2} x\), a direct consequence of our axially symmetric assumption.

Since the quadratic energy growth bound has been established in Theorem 1.4, the method in dimension 2 (see [18]) can be extended to our setting, which gives

**Lemma 7.1.** Suppose \(u\) is an axially symmetric solution of (1.1) with Morse index \(N\) in \(\mathbb{R}^3\). Then for any \(e \in \mathbb{R}^3\), there are at most \(2N\) nodal domains of \(u_e := e \cdot \nabla u\).

We first use this lemma to prove Theorem 1.5.

**Proof of Theorem 1.5.** If \(u\) is stable, by Lemma 7.1, \(u_z\) does not change sign. Then we can apply the main result in [2] to deduce the one dimensional symmetry of \(u\). Furthermore, by the axial symmetry, \(u(r, z) \equiv g(z - t)\) for some \(t \in \mathbb{R}\). □

Concerning solutions with Morse index 1, we first show

**Lemma 7.2.** An axially symmetric solution of (1.1) with Morse index 1 has at most three ends.

**Proof.** If the Morse index of \(u\) is 1, by Lemma 7.1 and Theorem 1.5, there are exactly two nodal domains of \(u_z\).

Assume there are at least 4 ends. Take 4 adjacent ones, \(\Gamma_\alpha, \alpha = 1, \cdots, 4\). Recall the notation \(\Omega_\alpha\) defined in Lemma 6.2. Assume \(u_z > 0\) in \(\Omega_1\) and \(\Omega_3\), \(u_z < 0\) in \(\Omega_2\) and \(\Omega_4\). Since \(\{u_z > 0\}\) is a connected set, there is a continuous curve connecting \(\Gamma_1\) and \(\Gamma_3\) in \(\{u_z > 0\}\), which gives a simple unbounded Jordan curve contained in \(\{u_z > 0\}\). Clearly \(\Omega_2\) and \(\Omega_4\) lies on different sides of this curve, therefore \(\{u_z < 0\}\) cannot be a connected set. This gives at least three nodal domains of \(u_z\), a contradiction. □

**Lemma 7.3.** Suppose \(u\) is an axially symmetric solution of (1.1) with Morse index 1. Then \(u_r > 0\) or \(u_r < 0\) strictly in \(\{r \neq 0\}\).
Proof. First note that \( \{u_r = 0\} \subset \{u_x = 0\} \). Hence it cannot have interior points. Assume by the contrary that there exist zero points of \( u_r \) in \( \{r \neq 0\} \). Then \( \{u_x = 0\} \cap \{r = 0\} \neq \emptyset \). Because most part of \{u_x = 0\} are smooth surfaces, \( \{u_x > 0\} \cap \{r = 0\} \neq \emptyset \) and \( \{u_x < 0\} \cap \{r = 0\} \neq \emptyset \). From this and the axial symmetry we deduce the existence of two open domains \( \Omega^\pm \) in the \((r, z)\) plane, where \( u_r > 0 \) in \( \Omega^+ \) and \( u_r < 0 \) in \( \Omega^- \). Viewing them as open domains in \( \mathbb{R}^3 \), then \( \Omega^+ \cap \{x_1 > 0\} \) and \( \Omega^- \cap \{x_1 < 0\} \) are two connected components of \( \{u_x > 0\} \), while \( \Omega^+ \cap \{x_1 < 0\} \) and \( \Omega^- \cap \{x_1 > 0\} \) are two connected components of \( \{u_x < 0\} \). Hence there are at least four nodal domains of \( u_x \), a contradiction with Lemma 7.1.

\( \square \)

**Proof of Theorem 1.6.** In view of Lemma 7.2, we only need to exclude the possibility of three ends.

By Lemma 7.3, we can assume \( u_r > 0 \) in \( \{r \neq 0\} \). Hence each connected component \( \Gamma_\alpha \) of \( \{u = 0\} \) is a graph in the \( r \)-direction. There are two cases:

**Type I.** \( \Gamma_\alpha \) is not disjoint from the \( z \) axis, hence it has the form \( \{r = f_\alpha(z)\} \) where \( f_\alpha \) is a function defined on an interval \([z_\alpha^-, z_\alpha^+]\) of the \( z \) axis and \( f_\alpha(z_\alpha^-) = 0 \);

**Type II.** \( \Gamma_\alpha \) is disjoint from the \( z \) axis, hence it has the form \( \{r = f_\alpha(z)\} \) where \( f_\alpha \) is a function defined on an open interval \((z_\alpha^-, z_\alpha^+)\) of the \( z \) axis.

For type I, we have \( \lim_{z \to z_\alpha^-} f_\alpha(z) = +\infty \), thus \( \Gamma_\alpha \) contributes one end. For Type II, we must have \( \lim_{z \to z_\alpha^+} f_\alpha(z) = +\infty \), thus \( \Gamma_\alpha \) contributes two ends. Since \( u \) has three ends, there are either three Type I components or one Type I plus one Type II components. Therefore \( u \) can change sign one time or three times on the \( z \)-axis.

**Case 1.** \( u \) changes sign three times on the \( z \)-axis.

In this case, there is an interval \((a^-, a^+)\) such that \( u(0, z) < 0 \) in \((a^-, a^+)\) and \( u(a^-) = u(a^+) = 0 \). Let \( \{z = f^\pm(r)\} \) be the connected components of \( \{u = 0\} \) emanating from \((0, a^\pm)\) respectively. Because \( u_r > 0 \), \( f^+(r) \) is decreasing in \( r \) and \( f^- \) is increasing. Hence

\[
\lim_{r \to +\infty} (f^+(r) - f^-(r)) \leq a^+ - a^-.
\]

This is a contradiction with Proposition 2.2.

**Case 2.** \( u \) changes sign one time on the \( z \)-axis.

Without loss of generality, assume \( u(0, 0) = 0, u(0, z) > 0 \) for \( z > 0 \) and \( u(0, z) < 0 \) for \( z < 0 \). There exists a connected component of \( \{u = 0\} \) emanating from \((0, 0)\), in the form \( \{z = f(r)\} \). As in Case 1, \( f \) is decreasing in \( r \). In particular, \( u > 0 \) in \( \{z > 0\} \). The other component of \( \{u = 0\} \) is Type II, which is represented by the graphs \( \{z = f^\pm(r)\} \) for two functions \( f^+ > f^- \) defined on \([R_*, +\infty)\) for some \( R_* > 0 \). Here \( f^+ \) is still increasing in \( r \). As in Case 1 we get

\[
\lim_{r \to +\infty} (f(r) - f^+(r)) < +\infty,
\]
a contradiction with Proposition 2.2 again. \( \square \)

**References**


