Last day

Linear transformations

Matrix representation of transformations.

If \( T: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \), then the matrix of the transformation is

\[
T = \begin{pmatrix}
1 & 1 \\
Tc_1 & Tc_2
\end{pmatrix}
\]

where \( e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) and \( e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) are the standard bases for \( \mathbb{R}^2 \).
**Composition of linear transformations.**

Let \( T: \mathbb{R}^n \rightarrow \mathbb{R}^p \) and \( S: \mathbb{R}^p \rightarrow \mathbb{R}^m \).

Then \( S(T(x)) \) is a composition of the two transformations and this \( T \) transformation is linear.

**T: \mathbb{R}^n \rightarrow \mathbb{R}^n \) and \( S: \mathbb{R}^n \rightarrow \mathbb{R}^n \)

In general, \( S(T(x)) \neq T(S(x)) \),

we can easily see this using matrix multiplication.

\[
S T \neq T S
\]

where \( S \) is the matrix for the transformation \( S \) and \( T \) is the matrix for the transformation \( T \).
Consider a system with 3 states, suppose a random walker can move from one location to another with a certain probability. Let $p_{ij}$ be the probability that a random walker at location $i$ will go to location $j$.

$$p_{3,3} \quad 3 \quad p_{3,2} \quad p_{3,1} \quad 2 \quad p_{2,2}$$

$$p_{1,3} \quad p_{1,1} \quad p_{1,2}$$
Consider location 2: suppose the random walker is at this location and he takes a step, there are 3 possibilities.

- he remains at location 2 with probability $P_{2,2}$
- he moves to 3 with probability $P_{3,2}$
- moves to 1 with probability $P_{1,2}$

and $P_{1,2} + P_{2,2} + P_{3,2} = 1$

In general,
$$P_{1,j} + P_{2,j} + P_{3,j} = \sum_{i=1}^{3} P_{i,j} = 1.$$  
for $j = 1, 2, 3$. 

Let $X_{n,j}$ be the probability that the random walker is at location $j$ at time $n$,

$$
\overrightarrow{X}_n = \begin{pmatrix}
X_{n,1} \\
X_{n,2} \\
X_{n,3}
\end{pmatrix}
$$

and if we have $\overrightarrow{X}_n$, then we can construct $\overrightarrow{X}_{n+1} = \begin{pmatrix}
X_{n+1,1} \\
X_{n+1,2} \\
X_{n+1,3}
\end{pmatrix}$

\[X_{n+1,1} = X_{n,1} p_{1,1} + X_{n,2} p_{1,2} + X_{n,3} p_{1,3}\]

\[X_{n+1,1} = X_{n,1} p_{2,1} + X_{n,2} p_{2,2} + X_{n,3} p_{2,3}\]

Then similarly,

\[X_{n+1,2} = X_{n,1} p_{3,1} + X_{n,2} p_{3,2} + X_{n,3} p_{3,3}\]

\[X_{n+1,3} = X_{n,1} p_{3,1} + X_{n,2} p_{3,2} + X_{n,3} p_{3,3}\]
We have a system of equations which can be written in matrix form as

\[
\begin{pmatrix}
X_{n+1, 1} \\
X_{n+1, 2} \\
X_{n+1, 3}
\end{pmatrix} =
\begin{pmatrix}
P_{11} & P_{12} & P_{13} \\
P_{21} & P_{22} & P_{23} \\
P_{31} & P_{32} & P_{33}
\end{pmatrix}
\begin{pmatrix}
X_{n, 1} \\
X_{n, 2} \\
X_{n, 3}
\end{pmatrix}
\]

\[ \overrightarrow{X_{n+1}} = P \overrightarrow{X_n} \]

Remark: The entries of each column of matrix \( P \) must sum to 1.

- Suppose \( \overrightarrow{X_0} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \) and \( \begin{pmatrix} 0.2 \\ 0.1 \\ 0.7 \end{pmatrix} \).

\[ \overrightarrow{X_1} = P \overrightarrow{X_0} \]
\[ \overrightarrow{X_2} = P \overrightarrow{X_1} = P (P \overrightarrow{X_0}) = P^2 \overrightarrow{X_0} \]
\[ \overrightarrow{X_k} = P^k \overrightarrow{X_0} \]
Remarks

1. The sum of the entries of $\overrightarrow{X}_t$ must equal 1.

2. This idea can be generalized to a system with arbitrary number of locations.

Example: Consider a random walk with 3 states as given below

Write down the matrix $\mathbf{P}$ of the system and find the probability that the random walker is at location 1 after 5 steps.

$$\overrightarrow{X}_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$
Solution

\[
\pi_{ij} = 1 \quad 2 \quad 3
\]

\[
P = \begin{pmatrix}
0 & Y_3 & Y_4 \\
0.5 & 0 & 0.314 \\
0.5 & 0.3 & 0
\end{pmatrix}
\]

Let us find \( \vec{X}_5 \)

\[
\vec{X}_5 = P^5 \vec{X}_0
\]

Using MATLAB,

\[
\vec{X}_5 = \begin{pmatrix}
0.2257 \\
0.3915 \\
0.3828
\end{pmatrix}
\]

The probability that the random walker will be at location 1 after 5 steps is 0.2257.
Example: Consider the system given below

\[ 0.3 \rightarrow 1 \rightarrow 2 \rightarrow 0.2 \]

\[ 0.1 \]

\[ 0.5 \]

\[ 0.3 \rightarrow 3 \rightarrow 4 \rightarrow 0.4 \]

Construct the matrix of the network and find the state of the network after 10 steps with \( \mathbf{X}_0 = \begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \end{pmatrix} \)

Solution

\[
\mathbf{P} = \begin{pmatrix}
0.3 & 0 & 0.1 & 0 & 0 \\
0.4 & 0.2 & 0.5 & 0 & 0 \\
0.3 & 0.8 & 0 & 0.6 & 0 \\
0 & 0 & 0.4 & 0.4 & 0 \\
1 & 2 & 3 & 4
\end{pmatrix}
\]
\[ \vec{X}_{10} = p^{10} \vec{X}_0 \]

and this gives

\[ \vec{X}_{10} = \begin{pmatrix} 0.0571 \\ 0.2485 \\ 0.3979 \\ 0.2865 \end{pmatrix} \]

**The Transpose**

Let \( A \) be an \( m \times n \) matrix, the transpose of \( A \) is the matrix whose rows are the columns of \( A \) (in the same order). It is denoted by \( A^T \).
Example:
Let \( A = \begin{pmatrix} 1 & 2 & 4 & 5 \\ 2 & 3 & 4 & 1 \\ 0 & 1 & 7 & 2 \\ 8 \end{pmatrix} \)

\( A^T = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 3 & 1 \\ 4 & 4 & 7 \\ 7 & 1 & 2 \\ 5 & 0 & 8 \end{pmatrix} \)

Note:
If \( A \) is a square matrix, and \( A^T = A \), then \( A \) is called a symmetric matrix.

Remarks:
1. Let \( A \) be an \( m \times n \) matrix. Then for any vector \( \vec{x} \in \mathbb{R}^n \) and \( \vec{y} \in \mathbb{R}^m \),
   \[ \vec{y} \cdot (A \vec{x}) = (A^T \vec{y}) \cdot \vec{x} \]
For two matrices $A$ and $B$,
\[(AB)^T = B^T A^T\]

**Matrix Inverses**

**Definition:**
Let $A$ be an $n \times n$ matrix. A matrix $B$ is called the inverse of $A$ if
\[BA = I\]

denoted by $B = A^{-1}$

Consider a system of equations given by
\[Ax = b\]

Let $B = A^{-1}$, and multiply the system by $B$ from the left.
\[B(Ax) = Bb\]
\[A^T(Ax) = A^Tb\]
\[(A^T A)x = A^Tb\] (we have used the associative property here)
\[ Ix = A^t b \]

\[ x = A^t b \]

**Remark**
- The inverse of a matrix is unique
- If \( b \) is the inverse of \( A \), then
  \[ AB = BA = I \]

**Relating inverse of a matrix to system of equations**

Let \( A \) be an \( n \times n \) matrix. The following statements are equivalent:

(i) \( A \) is invertible.
(ii) The system of equations \( A\hat{x} = b \) always has a unique solution.
(iii) The equation \( A\hat{x} = 0 \) has only the trivial solution.
(iv) The rank of \( A \) is \( n \).
The row echelon form of $A$ is of the form

\[
\begin{pmatrix}
\begin{array}{cccc}
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\end{array}
\end{pmatrix}
\]