Last day

* triple product

\[ \vec{a} = (a_1, a_2, a_3), \quad \vec{b} = (b_1, b_2, b_3), \quad \vec{c} = (c_1, c_2, c_3) \]

The triple product of \( \vec{a}, \vec{b}, \) and \( \vec{c} \) is

\[ \vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \]

* volume of a parallelepiped is \( |\vec{a} \cdot (\vec{b} \times \vec{c})| \)

* lines in 2D

* parametric form of lines in 2D

* line passes through the origin and in the direction of vector \( \vec{a} \)

\[ \vec{x} = t\vec{a} , \quad t \in \mathbb{R} \]

* line passes through the point \( \vec{q} \), and in the direction of vector \( \vec{a} \)

\[ \vec{x} = \vec{q} + t\vec{a} , \quad t \in \mathbb{R} \]
Pue! what is \( \vec{q} \) if the line passes through the origin?

We know that if \( \vec{a} \) is a line passing through the origin in the direction of \( \vec{a} \), then we have

\[
\vec{x}_1 = t \vec{a}
\]

If it passes through \( \vec{q} \) in the same direction,

\[
\vec{x}_2 = \vec{q} + t \vec{a}
\]

We want to find \( \vec{q} \) such that \( \vec{x}_1 = \vec{x}_2 \)

\[
\vec{x}_1 = \vec{x}_2
\]

\[
\Rightarrow t \vec{a} = \vec{q} + t \vec{a}
\]

\[
\Rightarrow \vec{q} \text{ must be } (0, 0) = \vec{0}
\]

**Note**

* The parametric form is not unique
* Each point on the line corresponds to a unique value of \( t \).
Example: Find the parametric form of the line that passes through the points \((-3, 1)\) and \((1, -2)\).

Solution:

\[A = (-3, 1), \quad B = (1, -2)\]

We know that each point on a line passing through point \(\vec{a}\) in the direction of \(\vec{a}\) is given by

\[\vec{r} = \vec{a} + t\vec{a}, \quad t \in \mathbb{R}\]

for some value of \(t\).

Let \(\vec{a} = \vec{B}\). Let us find \(\vec{a}\):

\[\vec{a} = \vec{B} - \vec{A} = (1, -2) - (-3, 1) = (4, -3)\]

Each point on the line can be written as

\[\vec{r} = (-3, 1) + t(4, -3), \quad \text{for some } t\]

If \(\vec{a} = (1, -2)\), then

\[\vec{r} = (1, -2) + t(4, -3), \quad \text{for some } t\]
Equation form of a line

Let us consider a line passing through the origin and in the direction of vector \( \vec{a} \).

\[ \vec{b} \text{ is orthogonal to } \vec{a}. \]

Let \( \vec{x} \) be a point on the line, then

\[ \vec{x} \cdot \vec{b} = 0 \]

Let \( \vec{x} = (x_1, x_2) \) and \( \vec{b} = (b_1, b_2) \)

Hence \( \vec{x} \cdot \vec{b} = (x_1, x_2) \cdot (b_1, b_2) \)

\[ \vec{x} \cdot \vec{b} = x_1 b_1 + x_2 b_2 \]

\[ \therefore \vec{x} \cdot \vec{b} = 0 \]

\[ \Rightarrow x_1 b_1 + x_2 b_2 = 0 \]

This is the equation form of a line in 2D passing through the origin in 2D.
Suppose the line passes through a point \( \vec{q} \).

\[
(\vec{x} - \vec{q}) \cdot \vec{b} = 0
\]
\[
\vec{x} \cdot \vec{b} - \vec{q} \cdot \vec{b} = 0
\]
\[
\vec{x} \cdot \vec{b} = \vec{q} \cdot \vec{b}
\]

If \( \vec{x} = (x_1, x_2) \) and \( \vec{b} = (b_1, b_2) \), then we have
\[
x_1 b_1 + x_2 b_2 = \vec{q} \cdot \vec{b}
\]

This is the equation form of a line passing through \( \vec{q} \) and orthogonal to the direction of \( \vec{b} \).
This is the equation form of a line passing through $\vec{q}$ and in the direction orthogonal to the direction of $\vec{b}$.

Example: Find the equation form for the line $\vec{q}$ given in parametric form as $(1,2) + t(1,2)$. 

Solution:
The equation of form of a line is given by $\vec{x} \cdot \vec{b} = \vec{q} \cdot \vec{b}$

We know that a line in parametric form is given by $\vec{x} = \vec{q} + t\vec{a}$.

$\vec{q} = (1,2), \vec{a} = (1,2)$

We need to find $\vec{b}$ and we know that $\vec{b}$ is orthogonal to $\vec{a}$.
How to find a vector orthogonal to another vector.

Let \( \vec{a} = (a_1, a_2) \)

we want to find a vector \( \vec{b} \) that is orthogonal to \( \vec{a} \).

Let \( \vec{b} = (x, y) \), since \( \vec{a} \) and \( \vec{b} \) are orthogonal,

then \( \vec{a} \cdot \vec{b} = 0 \)

\[ (a_1, a_2) \cdot (x, y) = 0 \]

\[ a_1 x + a_2 y = 0 \]

\[ x = \frac{-a_2 y}{a_1} \]

\[ x = \frac{-a_2}{a_1}, \quad y = a_1 \]

\[ \vec{b} = (-a_2, a_1) \]
continuation of example!

\[ \vec{a} = (1, 2) \quad \vec{b} = (4, 2) \]

\[ \vec{b} = (-2, 1) \]

\[ \vec{x} - \vec{b} = \vec{a} \cdot \vec{b} \]

let \( \vec{a} = (x_1, x_2) \)

then \( (x_1, x_2) \cdot (-2, 1) = (1, 2) \cdot (-2, 1) \)

\[-2x_1 + x_2 = -2 + 2 = 0 \]
\[-2x_1 + x_2 = 0 \]

This yields \( x_1 \) as an equation form of the line.
Example: Find the an equation form for the line whose parametric form is \[ \vec{x} = (0, 2) + t (2, 1) \]

Solution

The equation is given by
\[ \vec{x} \cdot \vec{b} = \vec{a} \cdot \vec{b} \]

From (1) \( \vec{a} = (2, 1) \), and \( \vec{q} = (0, 2) \).

We want to find \( \vec{b} \), \( \vec{b} \) is orthogonal to \( \vec{a} \).
\[ \vec{b} = (-1, 2) \]

\[ \vec{x} \cdot \vec{b} = \vec{a} \cdot \vec{b} \]

\[ (x_1, x_2) \cdot (-1, 2) = (0, 2) \cdot (-1, 2) \]

\[ -x_1 + 2x_2 = 4 \]
Example: Find a parametric form for the line whose equation form is given by

\[ x_1 + 4x_2 = 1 \]

Solution:

\[ x_1 + 4x_2 = 1 \]

We know that this is equivalent to

\[ \mathbf{x} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{b} \]

\[ \mathbf{x} = (x_1, x_2) \quad \text{and} \quad \mathbf{b} = (b_1, b_2) \]

\[ \mathbf{x} \cdot \mathbf{b} = x_1 b_1 + x_2 b_2 = x_1 + 4x_2 \]

\[ \Rightarrow b_1 = 1, \quad b_2 = 4 \]

\[ \Rightarrow \mathbf{b} = (1, 4) \]

\( \mathbf{b} \) is a point on the line.

Let \( x_1 = 2 \), substitute into \( \mathbf{1} \) to get \( x_2 \).

\[ 2 + 4x_2 = 1 \]

\[ 4x_2 = 1 - 2 = -1 \]

\[ x_2 = \frac{-1}{4} \]

\[ \mathbf{a} = \mathbf{b} - \mathbf{c} = (1, 4) - (2, \frac{1}{4}) = (1 - 2, 4 - \frac{1}{4}) = (-1, \frac{15}{4}) \]

\[ \Rightarrow \mathbf{x} = \mathbf{c} + t\mathbf{a} = (2, \frac{1}{4}) + t(-4, 1) \]
Determine if \(\left(\frac{9}{2}, \frac{3}{2}, 4\right)\) is on the line.

Solution

\[
\vec{x} = (4, 1, 3) + t(1, -1, 2)
\]

\[
\vec{x} = (4 + t, 1 - t, 3 + 2t)
\]

\[
\left(\frac{9}{2}, \frac{3}{2}, 4\right) = (4 + t, 1 - t, 3 + 2t)
\]

\[
\frac{9}{2} = 4 + t, \quad \frac{3}{2} = 1 - t, \quad 3 + 2t = 4
\]

\[
\frac{t}{2} = -\frac{1}{2}, \quad t = 1 - \frac{3}{2}, \quad 2t = 4 - 3 \quad t = \frac{1}{2}
\]

Since the value of the parameter is not unique, the point is not on the line.
### Line in 3D

* Parametric form

Let us consider a line passing through a point $\vec{q}$ and in the direction of vector $\vec{a}$. Then each point on the line is given by

$$\vec{X} = \vec{q} + t \vec{a}, \quad \text{for some } t.$$

**Example:** Find the parametric form of a line that passes through $(3,2,1)$ and $(4,1,3)$.

**Solution:**

We need $\vec{q}$ and $\vec{a}$.

$$\vec{a} = B - A = (4,1,3) - (3,2,1) = (1,-1,2)$$

Let $\vec{q} = (3,2,1)$, then the parametric form of the line is

$$\vec{X} = (3,2,1) + t (1,-1,2)$$

Another parametric form is

$$\vec{X} = (4,1,3) + t (1,-1,2)$$
Equation form of a line in 3D

Let \( \vec{b}_1 \) and \( \vec{b}_2 \) be non-collinear vectors in a plane.

Let \( \vec{x} \) be a point on the line.

Then \( \vec{x} \cdot \vec{b}_1 = 0 \)

and \( \vec{x} \cdot \vec{b}_2 = 0 \)

- The equation form of a line \( \vec{x} \) that passes through the origin is given by

\[
\begin{align*}
\vec{x} \cdot \vec{b}_1 &= 0 \\
\vec{x} \cdot \vec{b}_2 &= 0
\end{align*}
\]

If \( \vec{x} = (x_1, x_2, x_3) \), \( \vec{b}_1 = (b_{11}, b_{12}, b_{13}) \)

\( \vec{b}_2 = (b_{21}, b_{22}, b_{23}) \)

\[
\begin{align*}
\vec{x} \cdot \vec{b}_1 &= 0 \Rightarrow x_1 b_{11} + x_2 b_{12} + x_3 b_{13} = 0 \\
\vec{x} \cdot \vec{b}_2 &= 0 \Rightarrow x_1 b_{21} + x_2 b_{22} + x_3 b_{23} = 0
\end{align*}
\]
Suppose the line passes through point $\vec{q}$.

\[
\begin{align*}
(x - \vec{q}) \cdot \vec{b}_1 &= 0 \\
(x - \vec{q}) \cdot \vec{b}_2 &= 0
\end{align*}
\]