Last day

* LCR circuits

* Capacitors
  - Serves as source of voltage
  \[
  \frac{dV(t)}{dt} = -\frac{i}{C}
  \]
  where \( C \) - capacitance

* Inductors
  - Serves as source of current
  \[
  \frac{dI(t)}{dt} = -\frac{v}{L}, \quad L - inductance.
  \]

Usually we want to know how \( V(t) \) and \( I(t) \) evolve in time given a circuit.
- Derive differential equations for \( V(t) \) and \( I(t) \)
- Use the equations to construct a system of equations
- Use eigenanalysis to solve the system
Interpret the dynamics of $V(t)$ and $I(t)$ using the eigenvalues of the system.

**Revision**

**Example:** Find the real-valued general solution of the system given below.

$$\vec{X}(t) = A \vec{X}(t)$$

where

$$A = \begin{pmatrix} -1 & 2 \\ 1 & 0 \\ 2 & 1 \end{pmatrix}$$

For the eigenvalues,

\[
\begin{vmatrix}
-1 - \lambda & 2 \\
1 & -1 - \lambda \\
2 & 1 - \lambda \\
\end{vmatrix} = 0
\]

\[
(1 - \lambda) \begin{pmatrix} (1 - \lambda)^2 & +1 \\ -1 & (-1 - \lambda) + 2 & +1 & (1 - \lambda) \end{pmatrix} = 0
\]

\[
(1 - \lambda) \begin{pmatrix} (1 - \lambda)^2 & -1 & +2 \\ 1 - \lambda & \lambda^2 + 2 \lambda + 1 & 2 \lambda + 1 \end{pmatrix} = 0
\]
\[(1 - \lambda) \left( \lambda^2 - 2\lambda + 2 \right) = 0 \]
\[
\lambda_1 = 1, \quad \lambda^2 - 2\lambda + 2 = 0
\]
\[
\lambda_{2/3} = \frac{2 \pm \sqrt{4 - 8}}{2} = 1 \pm i
\]
\[
\lambda_2 = 1 + i
\]
\[
\lambda_3 = 1 - i
\]

For \( \lambda_1 = 1 \), \((A - \lambda_1 I) \hat{v} = 0\)

\[
\begin{pmatrix}
0 & -1 & 2 \\
-1 & 0 & 0 \\
-1 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
= \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
\]

\[
\begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & -1 & 2 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
1 \\
0 \\
0 \\
0
\end{pmatrix}
\]

\[-x_2 = -2x_3 \quad \Rightarrow \quad x_2 = 2x_3
\]
\[x_1 = 0\]

Take \( x_1 = 1 \),

\[
\hat{v}_1 = \begin{pmatrix}
0 \\
2 \\
1
\end{pmatrix}
\]
For $\lambda_2 = 1 + i$, $(A - \lambda_2 I) \mathbf{v}_2 = \mathbf{0}$

$$
\begin{align*}
\begin{pmatrix}
-1 & -1 & 2 \\
-i & -1 & 0 \\
-i & 0 & -i
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
\end{align*}
$$

$$
\begin{align*}
\begin{pmatrix}
-1 & -1 & 2 \\
-i & -1 & 0 \\
-i & 0 & -i
\end{pmatrix}
\sim
\begin{pmatrix}
1 & -1 & 0 \\
0 & -i & 0 \\
0 & -i & 1
\end{pmatrix}
\end{align*}
$$

$$
\begin{align*}
-x_2 = -x_3 \implies x_2 = x_3 \\
x_1 - c x_2 + 2i x_3 = 0 \\
x_1 = -2i x_3 + i x_3 = -2i x_3 + i x_3 = -i x_3
\end{align*}
$$

Take $x_3 = 1$, $x_1 = -i$, $x_2 = 1$

$$
\mathbf{v}_2 = \begin{pmatrix}
-i \\
i \\
i
\end{pmatrix}
$$

For $\lambda_3 = 1 - i$, $\mathbf{v}_3 = \begin{pmatrix}
i \\
i \\
i
\end{pmatrix}$
The general solution of the system is

\[ \vec{x}(t) = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2 + c_3 e^{\lambda_3 t} \vec{v}_3 \]

\[ = c_1 e^t \begin{pmatrix} 0 \\ -i \\ 1 \end{pmatrix} + c_2 e^{(1+i)t} \begin{pmatrix} -i \\ -1 \\ 1 \end{pmatrix} + c_3 e^{(1-i)t} \begin{pmatrix} i \\ 1 \\ 1 \end{pmatrix} \]

Consider

\[ c_1 e^t \begin{pmatrix} -i \\ -1 \\ 1 \end{pmatrix} + c_3 e^{(1-i)t} \begin{pmatrix} i \\ 1 \\ 1 \end{pmatrix} \]

\[ = e^t \left[ c_2 \begin{pmatrix} -i \\ -1 \\ 1 \end{pmatrix} e^t + c_3 e^{it} \begin{pmatrix} i \\ 1 \\ 1 \end{pmatrix} \right] \]

\[ = e^t \left[ c_2 \begin{pmatrix} -i \\ -1 \\ 1 \end{pmatrix} (\cos(t) + i \sin(t)) + c_3 e^{it} \begin{pmatrix} i \\ 1 \\ 1 \end{pmatrix} (\cos(t) - i \sin(t)) \right] \]

\[ = e^t \left[ -i c_2 \cos(t) - i^2 c_2 \sin(t) + c_2 i \cos(t) - i^2 c_3 \sin(t) \right. \]

\[ + c_3 \cos(t) + i c_3 \sin(t) + c_3 \cos(t) - i c_3 \sin(t) \]

\[ + c_2 \cos(t) + i c_2 \sin(t) + c_3 \cos(t) - i c_3 \sin(t) \]

\[ = e^t \begin{pmatrix} -c_2 \cos(t) + c_2 \sin(t) + i c_3 \cos(t) - c_3 \sin(t) & (c_2 + c_3) \cos(t) + i (c_3 - c_2) \cos(t) \\ (c_2 + c_3) \cos(t) + i (c_3 - c_2) \sin(t) & (c_2 + c_3) \cos(t) - i (c_3 - c_2) \sin(t) \end{pmatrix} \]
\[ x(t) = e^t \begin{bmatrix} (c_1 + c_2) & (\sin ct) \\ \cos ct & \cos ct \end{bmatrix} + \nu(c_3 - c_2) \begin{bmatrix} 2 \cos ct \\ -\sin ct \end{bmatrix} \]

let \( c_2 + c_3 = c_2 \) and \( c_3 - c_2 = c_3 \)

\[ x(t) = e^t \begin{bmatrix} c_2 (\sin ct) \\ \cos ct \end{bmatrix} + c_3 \begin{bmatrix} \cos ct \\ -\sin ct \end{bmatrix} \]

the real-valued general solution is

\[ x(t) = c_1 e^{\frac{c_2 t}{2}} + c_2 e^{\frac{c_3 t}{2}} \begin{bmatrix} \sin ct \\ \cos ct \end{bmatrix} + c_3 e^{\frac{c_3 t}{2}} \begin{bmatrix} \cos ct \\ -\sin ct \end{bmatrix} \]
Example: consider the random walk below

\[ \begin{array}{ccc}
  1 & \xrightarrow{\frac{1}{2}} & 2 \\
  \downarrow \frac{1}{3} & & \downarrow \frac{1}{3} \\
  \downarrow & & \uparrow \frac{1}{3} \\
  3 & & 2 \\
\end{array} \]

(a) construct the transition matrix \( P \)

(b) find \( \tilde{x}_n \) as \( n \to \infty \).

\[
P = \begin{pmatrix}
  0 & \frac{1}{3} & \frac{1}{3} \\
  \frac{1}{2} & 0 & \frac{1}{6} \\
  \frac{1}{2} & \frac{1}{3} & 0
\end{pmatrix}
\]

(c) if \( \lambda \) is an eigenvalue of \( P \), then

\[
| P - \lambda \mathbf{1} | = 0
\]

\[
\begin{vmatrix}
  -\lambda & \frac{1}{3} & \frac{1}{3} \\
  \frac{1}{2} & -\lambda & \frac{1}{6} \\
  \frac{1}{2} & \frac{1}{3} & -\lambda
\end{vmatrix} = 0
\]
\[ -\lambda \left( \lambda^2 - \frac{1}{2} \right) - \frac{\lambda}{3} \left( -\frac{7}{2} - \frac{3}{8} \right) + \frac{1}{4} \left( \frac{1}{2} + \frac{1}{3} \right) \]

\[ -\lambda \left( \lambda^2 - \frac{1}{2} \right) + \frac{3}{2} \lambda + \frac{5}{2\mu} = 0 \]

\[ -\lambda^3 + \frac{19}{2\mu} \lambda + \frac{5}{2\mu} = 0 \]

\[ 24\lambda^3 - 19\lambda - 5 = 0 \]

\[ \lambda = 1 \text{ is a solution} \]

\[ \Rightarrow \lambda - 1 = 0 \]

\[ \frac{24\lambda^3 - 19\lambda - 5}{(\lambda - 1)} \]

\[
\begin{array}{c|c}
\hline
24\lambda^3 - 19\lambda - 5 \\
\hline
24\lambda^3 - 24\lambda^2 \\
\hline
0 \quad 24\lambda^2 - 19\lambda \\
\hline
- (24\lambda^2 - 24\lambda) \\
\hline
0 \\
\hline
5\lambda - 5 \\
\hline
0 \\
\hline
0 \\
\hline
0 \\
\hline
0 \\
\hline
\end{array}
\]

\[ 5\lambda - 5 = 0 \]
\[(\lambda - 1) (24 \lambda^2 + 20 \lambda + 5) = 0 \]

\[\lambda = 1, \quad 24 \lambda^2 + 20 \lambda + 5 = 0\]

\[\Rightarrow \lambda_{2,3} = \frac{-12 \pm \sqrt{23}}{24} \]

Find the eigenvector corresponding to eigenvalue \(\lambda = 1\):

\[(\lambda - \lambda I) \mathbf{x} = 0\]

\[
\begin{pmatrix}
-1 & 4 & 4 \\
4 & -1 & 4 \\
4 & 4 & -1
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
\]

\[
\begin{pmatrix}
-1 & 4 & 4 \\
4 & -1 & 4 \\
4 & 4 & -1
\end{pmatrix}
\rightarrow
\begin{pmatrix}
-1 & 4 & 4 & 0 \\
4 & -1 & 4 & 0 \\
4 & 4 & -1 & 0
\end{pmatrix}
\rightarrow
\begin{pmatrix}
-1 & 4 & 4 & 0 \\
0 & -5/3 & 7/4 & 0 \\
0 & 5/3 & 7/4 & 0
\end{pmatrix}
\]

\[
\begin{pmatrix}
-1 & 4 & 4 & 0 \\
0 & -5/3 & 7/4 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

\[-\frac{5}{3} x_2 + \frac{7}{4} x_3 = 0 \quad \Rightarrow \quad x_2 = \frac{7}{20} x_3.\]
\[ -x_1 + \frac{1}{2} x_2 + \frac{1}{4} x_3 = 0 \]

\[ x_1 = \frac{1}{3} x_2 + \frac{1}{4} x_3 = \frac{1}{2} \quad x_3 = \frac{1}{2} \]

Take \( x_3 = \frac{1}{2} \), \( x_1 = 12 \), \( x_2 = 21 \)

\[
\overrightarrow{x} = \begin{pmatrix}
12 \\
21 \\
20
\end{pmatrix}
\]

Multiply by \( \frac{1}{53} \) to get a probability vector

\[
\overrightarrow{\hat{x}} = \begin{pmatrix}
\frac{12}{53} \\
\frac{21}{53} \\
\frac{20}{53}
\end{pmatrix}
\]

Since the matrix \( P \) has only one eigenvalue \( \lambda = 1 \), \( \overrightarrow{\hat{x}} \) as given in (\#) is the equilibrium probability of the network.