Correction to “Stability of parabolic Harnack inequalities on metric measure spaces”

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Dr. N. Kajino pointed us out that the proof of Lemma 3.3 in the paper [BBK] is inadequate, since there is no easy way to control the Green function $g^D_\lambda(x, y)$ near the boundary of $D$. Since there are also some other minor errors in Section 3, we have made a revision from page 499, line 6 to the end of Section 3. We thank Dr. Kajino for pointing out the error and for his comments on the revision.

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Let $Y$ be the process associated with the Dirichlet form $(\mathcal{E}, \mathcal{F})$. Let $G_\lambda$ be the $\lambda$-resolvent associated with the process $Y$; that is,

$$G_\lambda f(x) = \mathbb{E}^x \int_0^\infty e^{-\lambda t} f(Y_t) dt,$$

for bounded measurable $f$. Let $p_t(\cdot, \cdot)$ be the heat kernel of $Y$. Then the Green kernel of $G_\lambda$ is given by

$$g_\lambda(x, y) = \int_0^\infty e^{-\lambda t} p_t(x, y) dt.$$

We will use the Green kernel to build a cut-off function $\varphi$.

**Lemma 3.2.** Let $x_0 \in X$. Then there exist $\delta \in (0, 1)$ and $C_1 = C_1(\delta) > 0$ such that if $\lambda = c_0 \Psi(\delta R)^{-1}$, then

$$g_\lambda(x, y) \leq C_1 \frac{\Psi(R)}{V(x_0, R)}, \quad x \in B(x_0, R)^c, \; y \in B(x_0, \delta R), \quad (3.1)$$

$$g_\lambda(x, y) \geq 2C_1 \frac{\Psi(R)}{V(x_0, R)}, \quad x, y \in B(x_0, \delta R). \quad (3.2)$$

**Proof.** This follows easily from HK($\Psi$) by integration. □

**Lemma 3.3.** There exists $\theta > 0$ such that the following holds. Let $x_0 \in X$, $R > 0$, $x_1 \in B(x_0, R)$, and $\lambda \geq c \Psi(R)^{-1}$. Then

$$|g_\lambda(x_1, y) - g_\lambda(x_1, y')| \leq c_1 \left( \frac{d(y, y')}{R} \right)^\theta \frac{\Psi(R)}{V(x_0, R)} \quad \text{for } y, y' \in B(x_0, 2R)^c. \quad (3.3)$$
Proof. If \(d(y, y') \geq R/4\) then (3.3) follows immediately from (3.1). Otherwise we use the Hölder continuity of \(p_t(x_1, \cdot)\), which follows from PHI(\(\Psi\)) by a standard argument; see [BGK], Corollary 4.2. (Note that to handle small values of \(t\) we need to extend the function \(p_t(x_1, \cdot)\) from \((0, \infty) \times B(x_0, R)^c\) to \(\mathbb{R} \times B(x_0, R)^c\), by setting \(p_s(x_1, y) = 0\) for \(s < 0\).) Once we have the Hölder continuity of \(p_t(x_1, \cdot)\), integrating gives (3.3). \(\Box\)

The following lemma is given in [BH] Chapter I, Proposition I.4.1.1 when \(u, f \in \mathcal{F}\) are non-negative and bounded, and \(f \geq 0\). By a standard approximation argument, it can be proved for the unbounded case as well.

**Lemma 3.A.** For \(u \in \mathcal{F}\), let \(\Phi(u) = (u \vee 0) \wedge 1\). Then \(\Phi(u) \in \mathcal{F}\) and the following holds.

\[
\int_X f d\Gamma(\Phi(u), \Phi(u)) \leq \int_X f d\Gamma(u, u) \quad \forall f \in \mathcal{F} \text{ with } f \geq 0.
\]

Let \(\delta\) be as in Lemma 3.2, fix \(x_0 \in X\) and let \(B' = B(x_0, \delta R), B'' = B(x_0, \delta R/8), B = B(x_0, R), 2B = B(x_0, 2R)\). By Remark 2.6(2-3) it is enough to prove CS(\(\Psi\)) with a scale factor of \(\delta^{-1}\) rather than 2.

Let \(\lambda = c_0 \Psi(\delta R)^{-1}\) and define

\[
h := C_1 \Psi(R) \frac{V(x_0, \delta R/8)}{V(x_0, R)}.
\]

Integrating Lemma 3.2, we have the following:

\[
G_1B''(x) \leq h, \quad x \in B(x_0, R)^c, \quad (3.a)
\]

\[
G_1B''(x) \geq 2h, \quad x \in B(x_0, \delta R), \quad (3.b)
\]

\[
|G_1B''(x) - G_1B''(y)| \leq c_1 \left(\frac{d(x, y)}{R}\right)^{\theta} h, \quad x, y \in B(x_0, R) \setminus B(x_0, \delta R/2). \quad (3.c)
\]

Now define

\[
\varphi(x) = (2 \wedge h^{-1}G_1B''(x) - 1)^+ = \left(1 \wedge (h^{-1}G_1B''(x) - 1)^+\right) = \Phi(h^{-1}G_1B''(x) - 1).
\]

We need to make sure that \(\varphi \in \mathcal{F}\). For the purpose, let \(\hat{1}_{B(x_1, s)} \in \mathcal{F} \cap C_0\) be a function which is 1 inside \(B(x_1, s)\), between 0 and 1 in \(B(x_1, 2s) \setminus B(x_1, s)\) and 0 outside \(B(x_1, 2s)\). Then \(h^{-1}G_1B''(x) - 1 = h^{-1}G_1B''(x) - \hat{1}_{2B}(x)\) for \(x \in 2B\), so

\[
\varphi(x) = \Phi(h^{-1}G_1B''(x) - 1) = \Phi(h^{-1}G_1B''(x) - \hat{1}_{2B}(x)) \wedge \hat{1}_B \in \mathcal{F}.
\]

Using (3.a)–(3.c), it is easy to check that \(\varphi\) is a cut-off function for \(B' \subset B\) that satisfies Definition 2.5 (a)–(c). To complete the proof of CS(\(\Psi\)), we need to establish (2.5).
Proposition 3.4. Let $x_1 \in X$ and $f \in F$. Let $\delta$ be defined by Lemma 3.2 and let $I = B(x_1, \delta s)$ with $0 < s \leq R$ and $I^* = B(x_1, s)$. There exist $c_1, c_2 > 0$ such that for all $f \in F$,

\[
\int_I f^2d\Gamma(\varphi, \varphi) \leq c_1(s/R)^{2\theta}(\int_{I^*} d\Gamma(f, f) + c_2\Psi(s)\int_{I^*} f^2d\mu). \tag{3.4}
\]

Proof. Step 1. We first prove that there exists a cutoff function $\psi$ for $B' \subset B$, which we do not require to be continuous, such that

\[
\int_B f^2d\Gamma(\psi, \psi) \leq c_1\left(\int_X d\Gamma(f, f) + \Psi(R)^{-1}\int_X f^2d\mu\right). \tag{3.5}
\]

Let $D = B(x_0, R - \varepsilon)$ for some $\varepsilon > 0$ and define

\[
\mathcal{F}_D = \{f \in F : \tilde{f} = 0 \text{ q.e. on } X - D\}.
\]

Set

\[
\mathcal{E}_\lambda(f, g) = \mathcal{E}(f, g) + \lambda\int fg d\mu.
\]

Let $v = G^D_\lambda 1_{B'} \in F$. Note that

\[
v(x) \leq \int_B g^D(x, y)d\mu(y) \leq \frac{\mathbb{E}^x[\tau_D]}{\lambda} \leq c\Psi(R), \quad x \in D, \tag{3.5}
\]

by Theorem 2.15. By [FOT] Theorem 4.4.1, $v \in \mathcal{F}_D$ and is quasi-continuous. Further, since $Y$ is continuous, $v = 0$ on $D^c$. Let $f \in F$. Then

\[
\int_B f^2d\Gamma(v, v) \leq \int_X f^2d\Gamma(v, v) = \int_X d\Gamma(f^2v, v) - \int_X 2fvd\Gamma(f, v).
\]

Since $v \in \mathcal{F}_D$ we have $f^2v \in \mathcal{F}_D$, so by [FOT] Theorem 4.4.1,

\[
\int_X d\Gamma(f^2v, v) = \mathcal{E}(f^2v, G^D_\lambda 1_{B'}) \leq \mathcal{E}_\lambda(f^2v, G^D_\lambda 1_{B'}) = \int_X f^2v 1_{B'}d\mu \leq c\Psi(R)\int_{B'} f^2d\mu,
\]

where we used (3.5) in the last inequality. Using Cauchy-Schwarz and (3.5), we obtain

\[
\left|\int_X 2fvd\Gamma(f, v)\right| \leq c\left(\int_X v^2d\Gamma(f, f)\right)^{1/2}\left(\int_X f^2d\Gamma(v, v)\right)^{1/2} \leq c\Psi(R)\left(\int_B d\Gamma(f, f)\right)^{1/2}\left(\int_X f^2d\Gamma(v, v)\right)^{1/2}.
\]

So, writing $H = \int_X f^2d\Gamma(v, v)$, $J = \int_B d\Gamma(f, f)$, $K = \int_B f^2d\mu$, we have

\[
H \leq c\Psi(R)K + c\Psi(R)J^{1/2}H^{1/2},
\]

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from which it follows that $H \leq c\Psi(R)K + c\Psi(R)^2 I$. Let $\psi(x) = (v(x)/h) \wedge 1 = \Phi(v(x)/h)$. Computing similarly to Lemma 3.2 using [BGK] Theorem 3.1, $\psi(x) = 1$ for $x \in B(x_0, \delta R)$ so that $\psi$ is a cut-off function for $I \subset I^*$. Further, using Lemma 3.A, we have $\int_X f^2 d\Gamma(\psi, \psi) \leq h^{-2}H$. Thus (3.d) holds.

**Step 2.** In Step 2, we will consider the situation that either

$$I^* \subset B(x_0, \delta R) \quad (3.6)$$

or else

$$I^* \cap B(x_0, \delta R/2) = \emptyset. \quad (3.7)$$

Since $\varphi \equiv 1$ on $B(x_0, \delta R)$, (3.4) is clear if (3.6) holds. Thus, we consider when (3.7) holds. Let $\psi_s(x)$ be a cut-off function for $I \subset I^*$ given by Step 1. Let $\varphi_0(x) = h^{-1}G_\lambda 1_{B''}(x) \in \mathcal{F}$, $a_0 = \inf_{I^*} \varphi_0$ and $\varphi_1(x) = \varphi_0(x) - a_0 1_{I^*}(x) \in \mathcal{F}$. Note that $\varphi = \Phi(\varphi_1 + a_0 - 1)$ on $I^*$. By (3.c) we have

$$\varphi_1(x) \leq c(s/R)^\theta = L, \quad x \in I^*.$$

Let

$$A = \int_I f^2 d\Gamma(\varphi, \varphi),$$

$$D = \int_{I^*} d\Gamma(f, f) + \Psi(s)^{-1} \int_{I^*} f^2 d\mu,$$

$$F = \int_{I^*} f^2 \psi_s^2 d\Gamma(\varphi_1, \varphi_1).$$

By Lemma 3.A, we have

$$A \leq \int_I f^2 d\Gamma(\varphi_1, \varphi_1) \leq F = \int_{I^*} f^2 \psi_s^2 d\Gamma(\varphi_1, \varphi_0)$$

$$= \int_{I^*} d\Gamma(f^2 \psi_s^2 \varphi_1, \varphi_0) - \int_{I^*} \varphi_1 d\Gamma(f^2 \psi_s^2, \varphi_0). \quad (3.8)$$

For the first term in (3.8)

$$\int_{I^*} d\Gamma(f^2 \psi_s^2 \varphi_1, \varphi_0) = \int_X d\Gamma(f^2 \psi_s^2 \varphi_1, \varphi_0)$$

$$= \mathcal{E}_\lambda(f^2 \psi_s^2 \varphi_1, h^{-1}G_\lambda 1_{B''}) - \lambda \int_X f^2 \psi_s^2 \varphi_1 d\mu$$

$$\leq \mathcal{E}_\lambda(f^2 \psi_s^2 \varphi_1, h^{-1}G_\lambda 1_{B''}) = h^{-1} \int_{B''} f^2 \psi_s^2 \varphi_1 d\mu = 0.$$

Here we used the fact that $\varphi_1 \geq 0$ on $I^*$ and that the support of $\psi_s$ is in $I^*$, hence outside $B''$ (due to (3.7)).
The final term in (3.8) is handled, using the Leibniz and chain rules and Cauchy-Schwarz, as
\[
\left| \int_{I^*} \varphi_1 d\Gamma(f^2 \psi_s^2, \varphi_0) \right| \leq 2 \left| \int_{I^*} \varphi_1 f \psi_s^2 d\Gamma(f, \varphi_0) \right| + 2 \left| \int_{I^*} \varphi_1 f^2 \psi_s d\Gamma(\psi_s, \varphi_0) \right|
\]
\[
\leq c \left\{ \left( \int_{I^*} \psi_s^2 d\Gamma(f, f) \right)^{1/2} + \left( \int_{I^*} f^2 d\Gamma(\psi_s, \psi_s) \right)^{1/2} \right\} \left( \int_{I^*} \varphi_1 f^2 \psi_s^2 d\Gamma(\varphi_0, \varphi_0) \right)^{1/2}
\]
\[
\leq c' D^{1/2} LF^{1/2},
\]
where we used Step 1 in the final line. Thus we obtain \( A \leq F \leq cDL^2 \) so that (3.4) holds.

**Step 3.** We finally consider the general case. When either (3.6) or (3.7) holds, the result is already proved in Step 2. So assume that neither of them hold. Then \( I^* \) must intersect both \( B(x_0, \delta R/2) \) and \( B(x_0, \delta R)^c \), so \( s \geq \delta R/4 \). We use Lemma 2.3 to cover \( I \) with balls \( B_i = B(x_i, c_1 R) \), where \( c_1 \in (0, \delta/4) \) has been chosen small enough so that each \( B_i^* := B(x_i, c_1 R/\delta) \) satisfies at least one of (3.6) or (3.7). We can then apply (3.4) with \( I \) replaced by each ball \( B_i \): writing \( s' = c_1 R \) we have
\[
\int_{B_i} f^2 d\Gamma(\varphi, \varphi) \leq c_2(s'/R)^{2\alpha} \left( \int_{B_i^*} d\Gamma(f, f) + \Psi(s')^{-1} \int_{B_i^*} f^2 d\mu \right).
\]
We then sum over \( i \). Since no point of \( I^* \) is in more than \( L_0 \) (not depending on \( x_0 \) or \( R \)) of the \( B_i^* \), and \( c_1 s \leq s' \leq s \), we obtain (3.4) for \( I \).

\[\Box\]

**References**

