Differential equations

1. (15 points) Consider the following eigenvalue problem for $\Phi(x)$ with eigenvalue parameter $\lambda$:

$$x\Phi'' - \Phi' - \Phi = -x\lambda\Phi, \quad 1 < x < 2,$$
$$\Phi(1) = 0, \quad \Phi'(2) = -\Phi(2).$$

\hspace{1cm} (1)

(a) (4 points) Prove that any eigenvalue $\lambda$ for (1) must be real-valued.

(b) (4 points) Then, prove that any eigenvalue $\lambda$ for (1) must satisfy $\lambda > 0$.

(c) (3 points) State and derive the orthogonality relation for eigenfunctions of (1).

(d) (4 points) Finally, suppose that $f(x)$ satisfies the boundary value problem

$$xf'' - f' - f = 1, \quad 1 < x < 2,$$
$$f(1) = 0, \quad f'(2) = -f(2).$$

Find a formula for the coefficients $c_n$ in the eigenfunction representation $f(x) = \sum_{n=1}^{\infty} c_n \Phi_n(x)$ for the solution to (2). Here, $\Phi_n(x)$ for $n \geq 1$ are the eigenfunctions of (1).

2. (15 points) Let $\omega > 0$ be a real-valued constant, and consider the fourth-order initial-value problem, defined on $t \geq 0$, for $y(t)$

$$y'''' - y = 4\cos(\omega t).$$

\hspace{1cm} (4)

(a) (5 points) For $\omega \neq 1$, find the general solution to (4) in terms of arbitrary coefficients.

(b) (4 points) Consider (4) with $\omega \neq 1$ with the initial values $y(0) = A$ and $y'(0) = y''(0) = y'''(0) = 0$. Determine a formula for $A$ in terms of $\omega$ so that $y(t)$ is bounded as $t \to \infty$.

(c) (3 points) Find the particular solution to (4) when $\omega = 1$.

(d) (3 points) Finally, for $\omega \neq 1$ consider the modified initial value problem on $t > 0$

$$y'''' + y = 4\cos(\omega t), \quad \text{with} \quad y(0) = A, \quad y'(0) = y''(0) = y'''(0) = 0.$$ 

\hspace{1cm} (5)

Is there a value of $A$ for which $y(t)$ is bounded as $t \to \infty$? Explain your answer clearly.

3. (15 points) Consider the diffusion problem for $u(r, \theta, t)$ in a 2-D disk of radius $a$ with an inflow/outflow flux boundary condition modeled by

$$u_t = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}, \quad 0 \leq r \leq a, \quad 0 \leq \theta \leq 2\pi, \quad t \geq 0,$$
$$u_r(a, \theta, t) = f(\theta), \quad u \text{ bounded as } r \to 0, \quad u \text{ and } u_\theta \text{ are } 2\pi \text{ periodic in } \theta, \quad u(r, \theta, 0) = g(r, \theta).$$

(a) (3 points) Write the problem that the \textbf{steady-state solution} $U(r, \theta)$ would satisfy. Prove that such a steady-state solution $U(r, \theta)$ does not exist when $\int_0^{2\pi} f(\theta) \, d\theta \neq 0$.

(b) (8 points) Assume that $\int_0^{2\pi} f(\theta) \, d\theta = 0$. Calculate an integral representation for the \textbf{steady state solution} $U(r, \theta)$ by summing an appropriate eigenfunction expansion.

(c) (4 points) Assume that $\int_0^{2\pi} f(\theta) \, d\theta \neq 0$. Calculate an expression for the spatial average of $u$ over the disk, i.e. for $(\pi a^2)^{-1} \int_0^{2\pi} \int_0^a u \, r \, dr \, d\theta$, and interpret the effect on this average of the net boundary flux $\int_0^{2\pi} f(\theta) \, d\theta$. 
Linear Algebra

4. (15 points) Consider the following statements. Either prove the statements are true for all matrices with real entries or provide a counter-example. Note that an orthogonal matrix is square with nonzero, mutually orthogonal columns. \( A^T \) denotes the transpose of \( A \).

(a) (3 points) The product of two \( n \times n \) orthogonal matrices is invertible.
(b) (3 points) The difference between two distinct \( n \times n \) orthogonal matrices cannot be singular.
(c) (3 points) The product of a symmetric matrix and a diagonal matrix is always symmetric.
(d) (3 points) The Range of an \( n \times n \) matrix is perpendicular to its Nullspace.
(e) (3 points) If \( A \) is an \( n \times n \) matrix with \( n \) odd and \( A = -A^T \) then \( A \) must be singular.

5. (15 points) Consider real matrices with the block form

\[
C = \begin{bmatrix} A & B \\ B^T & 0 \end{bmatrix}
\]

where \( A \) is a symmetric square matrix, \( B^T \) denotes the transpose of \( B \) and \( B \) is not necessarily square. The bottom right block is a square matrix of zeros.

(a) (5 points) Show that \( C \) is singular if the number of columns of \( B \) is strictly larger than the number of rows.
(b) (10 points) Show that if \( A \) is strictly positive definite, then \( C \) is nonsingular iff the columns of \( B \) are linearly independent.

6. (15 points) Let \( I \in \mathbb{R}^{N,N} \) be the \( N \times N \) dimensional identity matrix, where \( N \geq 2 \) is an integer, and let \( u \in \mathbb{R}^N \) and \( v \in \mathbb{R}^N \) be any two distinct vectors each with Euclidean length one. Define the matrix \( A \) by

\[
A = I - uv^T.
\]

(a) (5 points) Calculate all the eigenvalues and eigenvectors of \( A \)
(b) (3 points) Prove that \( A \) is nonsingular and calculate \( \det(A) \).
(c) (4 points) Derive an explicit formula for \( A^{-1} \).
(d) (3 points) Let \( I \in \mathbb{R}^{N,N} \) for \( N \geq 2 \) be the identity matrix and define \( e \in \mathbb{R}^N \equiv (1, \ldots, 1)^T \) and \( e_1 \in \mathbb{R}^N \equiv (1,0,0,\ldots,0)^T \). Prove that the following linear system

\[
\left( I - \frac{1}{N}ee^T \right) x = e_1,
\]

has no solution. Next, if \( e_1 \) is replaced by an arbitrary vector \( b \), what is the condition on \( b \) for this problem to have a solution?