1. (15 points) Consider the following problem for the two-component vector $y(t)$ satisfying the linear system of ODEs

$$y'' + \begin{pmatrix} 5 & -1 \\ 2 & 2 \end{pmatrix} y = F \cos(\omega t) \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$ 

Here $F > 0$ and $\omega > 0$ are constants.

(a) Determine in as explicit a form as you can the general solution to the homogeneous problem where $F = 0$.

(b) Calculate a particular solution for $F > 0$ and for any $\omega$ with $\omega \neq \sqrt{3}$ and $\omega \neq 2$.

(c) Calculate a particular solution for $F > 0$ and $\omega = 2$. (Hint: this is a resonance case).

(d) When $\omega = 3$ and $F = 1$, find an explicit solution to the initial value problem with initial condition

$$y(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad y'(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

2. (15 points) Let $c_0$ and $c$ be positive constants with $0 < c_0 < c$. Calculate the solution $u(x, t)$ to each of the following two PDE wave problems:

(a) The one-way wave equation:

$$u_t + c u_x = 0, \quad c_0 t \leq x < \infty, \quad t \geq 0,$$

$$u(x, 0) = f(x), \quad u(c_0 t, t) = h(t).$$

(b) The wave equation:

$$u_{tt} = c^2 u_{xx}, \quad c_0 t \leq x < \infty, \quad t \geq 0,$$

$$u(x, 0) = f(x), \quad u_t(x, 0) = 0, \quad u(c_0 t, t) = h(t).$$

(c) Briefly explain (three clear sentences is sufficient here) the qualitative differences between the solutions to these two problems.

3. (15 points) Let $u(r, \theta)$ satisfy Laplace’s equation in polar coordinates inside the disk $0 \leq r \leq R$ with $0 \leq \theta \leq 2\pi$, satisfying

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0, \quad 0 \leq r \leq R, \quad 0 \leq \theta \leq 2\pi,$$

$$u(R, \theta) = f(\theta); \quad u \text{ bounded as } r \to 0; \quad u, u_\theta, \quad 2\pi \text{ periodic in } \theta.$$

(a) Determine an explicit solution for $u$ when $f(\theta) = 2 \cos^2(2\theta)$.

(b) For an arbitrary smooth $2\pi$ periodic $f(\theta)$ derive an infinite series representation for $u(r, \theta)$ in terms of the angular eigenfunctions.

(c) Sum the infinite series in (ii) to obtain the well-known Poisson’s integral formula for $u(r, \theta)$. (Hint: you will need to calculate an infinite sum of the form $\sum_{n=1}^{\infty} \rho^n \cos(n\theta)$ for $0 < \rho < 1$.)
4. (15 points) Consider the matrix

\[ A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 3 & 4 \end{bmatrix} . \]

(a) (3 points) What is the rank of \( A \)?

(b) (4 points) Find a basis for the nullspace of \( A^T \) (the transpose of \( A \)).

(c) (6 points) Determine all values of \( w \) for which the system

\[
\begin{align*}
    x + y &= -1 \\
    x + 2y &= w \\
    3x + 4y &= 0
\end{align*}
\]

has a solution and find one.

(d) (2 points) Is the solution in (c) above unique?

5. (15 points) Consider the sequence \( \{v^n\}_{n=0}^\infty \) of vectors in \( \mathbb{R}^2 \) defined by given \( v^0 \) and the recurrence relationship

\[ Av^{n+1} = Bv^n \]

where

\[ A = \begin{bmatrix} 1 & -k \\ k & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & k \\ -k & 1 \end{bmatrix} \]

and \( k > 0 \) is a given parameter.

(a) (8 points) Rewrite the recurrence relationship in the form

\[ v^{n+1} = C v^n \]

with \( C \) a matrix.

(b) (7 points) Show that \( \|v^n\| = \|v^0\| \) for all \( n \) where \( \| \cdot \| \) is the standard Euclidean norm.

6. (15 points) (a) (7 points) Prove that

\[ e^A := I + A + A^2/2! + \cdots + A^n/n! + \cdots \]

converges at every index for any square matrix. Here, \( I \) is the identity matrix.

(b) (4 points) Find \( e^A \) when

\[ A = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} \]

(c) (4 points) Find \( e^A \) when

\[ A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \]