Key for the 2002 Calculus Challenge Exam

Note: There is no attempt here to describe all possible correct answers. Students sitting the calculus challenge examination will have used a variety of texts and been exposed to a variety of teaching styles.

For the examiners, apart from the accuracy of the answers, the crucial test is whether the student has made clear the principles and/or method being used and whether those principles and/or method are sound.

Marks are not deducted for sufficiently trivial errors, e.g., inadvertently dropping a sign.

1. Compute the following limits.

\[ \text{ANSWER: } -5 \]

**JUSTIFY YOUR ANSWER**

Note that \( \frac{t^2 - t - 6}{t^2 + 5t + 6} = \frac{(t + 2)(t - 3)}{(t + 2)(t + 3)} = \frac{t - 3}{t + 3} \) whenever \( t \neq -2 \). Therefore

\[
\lim_{t \to -2} \frac{t^2 - t - 6}{t^2 + 5t + 6} = \lim_{t \to -2} \frac{t - 3}{t + 3} = \frac{(\lim_{t \to -2} t) - 3}{(\lim_{t \to -2} t) + 3} = \frac{-2 - 3}{-2 + 3} = -5.
\]

Instead of using the limit laws to evaluate \( \lim_{t \to -2} \frac{t - 3}{t + 3} \), one may use the fact that a rational function is continuous, i.e., continuous at each point of its domain.

\[ \text{ANSWER: } 5 \]

**JUSTIFY YOUR ANSWER**

From the limit laws the given limit is equal to

\[
\left( \lim_{x \to 0^+} \frac{\sin x}{x} \right) \left( 3 \lim_{x \to 0^+} e^{-1/x} + 5 \lim_{x \to 0^+} e^x \right)
\]

provided the three limits in the line above exist. We may take \( \lim_{x \to 0^+} (\sin x)/x = 1 \) as known. (A second way of looking at this limit is via l’Hospital’s rule. A third way is to notice that, since \( (d/dx) \sin x = \cos x \), we have \( \lim_{x \to 0} (\sin h)/h = \cos 0 = 1 \).) Now \( e^x \) is continuous so \( \lim_{x \to 0^+} e^x = e^0 = 1 \). Finally, as \( x \to 0^+ \), \( 1/x \to \infty \), whence \( e^{1/x} \to \infty \) and so \( e^{-1/x} \to 0 \). Thus the given limit evaluates to \( 1 \cdot (0 + 5) = 5 \).
2. (a) Find the asymptotes of \( y = \left( \frac{x}{x-1} \right)^2 \) and justify your answer.

**ANSWER:**
\[
\begin{align*}
y &= 1 \\
x &= 1
\end{align*}
\]

**EXPLANATION**

The following is an acceptable explanation: \([x/(x-1)]^2\) is defined except at \(x = 1\). Also,
\[
\lim_{x \to -\infty} \left( \frac{x}{x-1} \right)^2 = \lim_{x \to \infty} \left( \frac{x}{x-1} \right)^2 = 1 \quad \text{and} \quad \lim_{x \to 1^+} \left( \frac{x}{x-1} \right)^2 = \lim_{x \to 1^-} \left( \frac{x}{x-1} \right)^2 = \infty.
\]

2. (b) Where does the curve \( y = \left( \frac{x}{x-1} \right)^2 \) cross its horizontal asymptote?

**ANSWER:** \((1/2, 1)\)

**EXPLANATION**

We have to solve the equations: \( y = 1 \) and \( y = [x/(x-1)]^2 \). Eliminating \( y \), we have \( x = \pm(x-1) \). The only solution is \( x = 1/2 \), which gives \( y = 1 \).

3. (a) Find \( \frac{dv}{du} \) when \( v = \sqrt{\frac{\tan u}{1 + \tan u}} \).

**ANSWER:**
\[
\frac{dv}{du} = \frac{\sec^2 u}{2 (1 + \tan u)^{3/2} (\tan u)^{1/2}}
\]

**SHOW YOUR WORK**

Using the chain rule and the quotient rule, we have:
\[
\frac{dv}{du} = \frac{1}{2} \sqrt{\frac{1 + \tan u}{\tan u}} \frac{d}{du} \left( \frac{\tan u}{1 + \tan u} \right) = \frac{1}{2} \left( \frac{1 + \tan u}{\tan u} \right)^{1/2} \frac{(1 + \tan u) \sec^2 u - \tan u \sec^2 u}{(1 + \tan u)^2}
\]

3. (b) Let \( a \) be a constant and \( f(x) = \sin(ax) \). Find the 97-th derivative, \( f^{(97)}(x) \), of the function \( f(x) \).

**ANSWER:** \( a^{97} \cos ax \)

**SHOW YOUR WORK**

We have: \( f(x) = \sin(ax) \), \( f'(x) = a \cos(ax) \), \( f''(x) = -a^2 \sin(ax) \), \( f^{(3)}(x) = -a^3 \cos(ax) \), \( f^{(4)}(x) = a^4 \sin(ax) \), \ldots. There is a clear pattern from which we deduce \( f^{(96)}(x) = a^{96} \sin(ax) \).
4. (a) Find the general antiderivative of \((9 - 4x^2)^{-1/2}\).

\[
\frac{1}{2} \sin^{-1} \left( \frac{2x}{3} \right) + C.
\]

**SHOW YOUR WORK:** Using the substitution \(u = (2x)/3\), we have:

\[
\int (9 - 4x^2)^{-1/2} \, dx = \int (9 - 4(3u/2)^2)^{-1/2} \, (3/2) \, du = \frac{1}{2} \int (1 - u^2)^{1/2} \, du = \frac{1}{2} \sin^{-1} u + C.
\]

(b) It is given that \(f'(x) = 2^x + x^2\) and \(f(0) = 0\).

Find \(f(x)\).

**SHOW YOUR WORK:** Writing \(2^x\) as \(e^{x \ln 2}\) we see that the antiderivative of \(2^x + x^2\) is \(\left( e^{x \ln 2} / \ln 2 \right) + x^3/3\). By the evaluation theorem,

\[
f(x) - f(0) = \int_0^x f'(t) \, dt = \left[ \frac{2^t}{\ln 2} + \frac{t^3}{3} \right]_0 = \frac{2^x - 1}{\ln 2} + \frac{x^3}{3}.
\]

5. Use the definition of derivative (and not the product rule) to show that, if \(f(x)\) is differentiable at \(x = c\) and \(g(x) = xf(x)\), then \(g'(c)\) exists and \(g'(c) = f(c) + cf'(c)\).

**ANSWER:** Using the definition of derivative we have

\[
g'(c) = \lim_{h \to 0} \frac{g(c + h) - g(c)}{h} = \lim_{h \to 0} \frac{(c + h)f(c + h) - cf(c)}{h}
\]

\[
= \lim_{h \to 0} \left[ c \left( \frac{f(c + h) - f(c)}{h} \right) + f(c + h) \right] = c \lim_{h \to 0} \left( \frac{f(c + h) - f(c)}{h} \right) + \lim_{h \to 0} f(c + h)
\]

\[
= cf'(c) + f(c).
\]

Note that \(\lim_{h \to 0} f(c + h) = f(c)\) because differentiability of \(f\) at \(c\) (which is assumed) implies continuity at \(c\).

6. For what value of \(k\) is the function

\[
h(x) = \begin{cases} 
2x + 3 & \text{if } x \leq 1 \\
 k - 1 & \text{if } x > 1
\end{cases}
\]

continuous?

**JUSTIFY YOUR ANSWER:** Since \(2x + 3\) and \(k - 1\) are continuous functions, \(h(x)\) is continuous at \(x = c\) for all \(c \in (-\infty, 1) \cup (1, \infty)\) and continuous on the left at \(x = 1\). Further, \(\lim_{x \to 1^-} h(x) = k - 1\) is equal to \(h(1) = 5\) if and only if \(k = 6\).
7. (a) Express \( \frac{dy}{dx} \) as a function of \( x \), when \( y = \left( \frac{x^7 \cos x}{7x^2 \sqrt{1 + x^2}} \right) \).

\[
\frac{dy}{dx} = \left( \frac{x^7 \cos x}{7x^2 \sqrt{1 + x^2}} \right) \left( \frac{7}{x} - \tan x - \ln 7 - \frac{x}{1 + x^2} \right)
\]

SHOW YOUR WORK: Taking natural logarithms and differentiating, we get

\[
\frac{1}{y} \frac{dy}{dx} = \frac{d}{dx} \left( \ln x + \ln \cos x - x \ln 7 - \frac{1}{2} \ln(1 + x^2) \right) = \frac{7}{x} - \tan x - \ln 7 - \frac{x}{1 + x^2}.
\]

Although the line above is only valid for values of \( x \) such that \( x, \cos x > 0 \), the resulting formula is valid for all \( x \neq 0 \) such that \( \tan x \) is defined.

(b) Express \( \frac{dy}{dx} \) as a function of \( x \), when \( y = x^{\ln x} \).

\[
\frac{dy}{dx} = 2(\ln x)x^{(\ln x)-1}
\]

SHOW YOUR WORK: We have

\[
\frac{dy}{dx} = \frac{d}{dx} \left( (e^{\ln x})^{\ln x} \right) = \frac{d}{dx} \left( e^{(\ln x)^2} \right) = e^{(\ln x)^2} \frac{d}{dx}(\ln x)^2 = 2(\ln x)x^{(\ln x)-1}.
\]
8. A curve has the equation $\sin(x + y) = xe^y$.

[2] (a) Show that $(0, \pi)$ is on the curve.

**ANSWER:** We just observe that $\sin(0 + \pi) = 0 = 0 \cdot e^\pi$.

[4] (b) Find the equation of the line tangent to the curve at $(0, \pi)$.

**ANSWER:**

$$y + (1 + e^\pi)x = \pi$$

**SHOW YOUR WORK:** By implicit differentiation,

$$\cos(x + y) \left( 1 + \frac{dy}{dx} \right) = e^y + xe^y \frac{dy}{dx}.$$ 

It follows that $\left( \frac{dy}{dx} \right)_{x=0,y=\pi} = -1 - e^\pi$. This allows us to write down the equation of the tangent using the point-slope form of the equation of a line.

[4] (c) A point moves along the curve so that at $(0, \pi)$ its $x$-coordinate is increasing at a rate of 3 units/sec. How fast is its $y$-coordinate changing at $(0, \pi)$?

**ANSWER:** decreasing by $3(1 + e^\pi)$ units per sec

**SHOW YOUR WORK:** In general, $\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt}$. Thus

$$\left( \frac{dy}{dt} \right)_{x=0,y=\pi} = (-1 - e^\pi) \left( \frac{dx}{dt} \right)_{x=0,y=\pi} = -3(1 + e^\pi).$$
9. Let \( f(x) = e^{x-2} + x^3 - 2 \).

(a) Use the derivative of \( f \) to explain why the equation \( f(x) = 0 \) has at most one solution.

EXPLANATION: Since \( f'(x) = e^{x-2} + 3x^2 > 0 \) for all \( x \), the function \( f \) is strictly increasing.

(b) Explain why \( f(x) = 0 \) has a solution in the interval \((1, 2)\).

EXPLANATION: Note that \( f \) is continuous, \( f(1) = 1 - (1/e) < 0 \), and \( f(2) = 1 + 8 - 2 = 7 > 0 \). By the intermediate value theorem, \( f \) has a zero in \((1, 2)\).

(c) Newton's method with an initial estimate of 2 is used to find an approximate value for the solution of \( f(x) = 0 \).

What is the next estimate?

SHOW YOUR WORK: The next estimate is \( 2 - (f(2)/f'(2)) = 2 - (7/13) = 19/13 \). This is the \( x \)-coordinate of the point in which the tangent to \( y = f(x) \) at \((2, f(2))\) meets \( y = 0 \).

10. A particle moves along the \( x \)-axis with velocity \( \frac{1}{1 + t^2} \) at time \( t \). If it passes the point \( \pi/6 \) at time \( t = 1 \), what is its acceleration when it passes the point \( \pi/4 \)?

ANSWER: \(-\sqrt{3}/8\)

SHOW YOUR WORK: We are given \( \frac{dx}{dt} = \frac{1}{1 + t^2} \). Taking antiderivatives, we get \( x = \tan^{-1} t + C \), where \( C \) is a constant. Since \( x(1) = \pi/6 \), we see that \( \pi/6 = (\pi/4) + C \). So \( C = -\pi/12 \). Letting \( x = \pi/4 \), we get \( \pi/4 = \tan^{-1} t - (\pi/12) \). So \( \tan^{-1} t = \pi/3 \), which means \( t = \sqrt{3} \) when \( x = \pi/4 \).

The acceleration is given by

\[
\frac{d^2x}{dt^2} = \frac{d}{dt} \left( \frac{1}{1 + t^2} \right) = \frac{-2t}{(1 + t^2)^2}.
\]

Substituting \( t = \sqrt{3} \), we get \(-\sqrt{3}/8\) for the acceleration.
11. A rectangle has two adjacent vertices \((-t, 0)\) and \((t, 0)\) on the \(x\)-axis and the other two on the parabola \(y = k - x^2\), where \(k > 0\). For each \(k\) there exists \(t > 0\) which maximizes the area of the resulting rectangle.

Find \(k\) such that the rectangle of maximum area is a square.

**ANSWER:**

\[k = 3\]

**SHOW YOUR WORK:** From the figure, the area of the rectangle is given by:

\[A = 2t(k - t^2).\]

Now \(dA/dt = 2k - 6t^2\) is 0 when \(t = \pm \sqrt{k/3}\). Since \(dA/dt > 0\) for \(t \in (0, \sqrt{k/3})\) and \(dA/dt < 0\) for \(t \in (\sqrt{k/3}, \sqrt{k})\), \(t = \sqrt{k/3}\) gives the maximum area. For the resulting rectangle to be a square, we need

\[2\sqrt{k/3} = 2t = k - t^2 = k - (\sqrt{k/3})^2 = 2k/3.\]

The only solution is \(k = 3\) because \(k\) is constrained to satisfy \(k > 0\).

12. Find the line \(y = mx\) through the origin, with positive slope, which together with the fragment of the parabola

\[y = x^2 \quad (0 \leq x \leq m)\]

encloses a region of area \(4/3\).

**ANSWER:**

\[y = 2x\]

**SHOW YOUR WORK:** The area below \(y = x^2\) from \(x = 0\) to \(m\) is found to be \(\int_0^m x^2 \, dx = m^3/3\). Thus the area between \(y = m^2x\) and the parabola is \((m^3/2) - (m^3/3) = m^3/6\). For \(m^3/6 = 4/3\) we need \(m = 2\).
13. A bacteria-infested swimming pool was chemically treated this morning, and since then, the bacteria count has been decreasing at rate proportional to the count itself.

An hour ago, the count was a third of what it was two hours ago. For safety, the count must be \( \leq 1\% \) of what it is now.

When will that be?

ANSWER: In about 4.2 hours

SHOW YOUR WORK: Let \( C(t) \) be the bacteria count at time \( t \). It is given that \( \frac{dC}{dt} = kC \), where \( k \) is a constant. This equation can be rewritten as \( \frac{d}{dt} \ln C = k \). Taking antiderivatives gives \( \ln C = kt + b \), where \( b \) is a constant. Hence \( C = Be^{kt} \), where \( B = e^b \). Let \( t \) be measured in hours from "now". Then \( C(-1) = (1/3)C(-2) \) tells us that \( e^k = 1/3 \). Clearly, \( C(0) = B \). So for the count to be \( \leq B/100 \) we need \( e^{kt} < 100 \), which is \((1/3)^t \leq 1/100\). Rearranging we get \( 3^t \geq 100 \), which means \( t \geq (\ln 100)/(\ln 3) = \log_3 100 \approx 4.2 \).

14. Let \( f(x) = 2x^4 - 3x^3 + 5x^2 - 3x + 2 \).

(a) Find the line tangent to \( y = f(x) \) at \( x = 0 \).

ANSWER: \( y + 3x = 2 \)

SHOW YOUR WORK: The point on \( y = f(x) \) at \( x = 0 \) is \( (0, f(0)) = (0, 2) \). By inspection, \( f'(0) = -3 \). Hence the tangent at \( (0, 2) \) is \( y + 3x = 2 \).

(b) Show that the line found in (a) intersects \( y = f(x) \) only at \( x = 0 \).

EXPLANATION: The graphs of \( y = f(x) \) and \( y + 3x = 2 \) intersect where \( f(x) = 3x + 2 \). This gives 
\[
2x^4 - 3x^3 + 5x^2 - 3x + 2 = -3x + 2.
\]
Simplifying, we get
\[
x^2(2x^2 - 3x + 5) = 0.
\]
Since the discriminant of \( 2x^2 - 3x + 5 \) is \((-3)^2 - 4 \cdot 2 \cdot 5 = -31 < 0 \), the quadratic has no real solutions. Thus \( x = 0 \) is the only solution.

(c) Use a linear approximation to estimate \( f(0.01) \).

ANSWER: \( f(.01) \approx 1.97 \)

SHOW YOUR WORK: The linear approximation is \( f(0) + (.01)f'(0) = 2 - 3(.01) \).
(d) Let a real number $a$ be given as well as the exact value of $f(a)$. Now suppose that a linear approximation is used to estimate $f(a + 0.01)$. Show that the estimate will be an underestimate whatever the value of $a$.

EXPLAIN: The intuition is that the estimate will always be an underestimate provided $f'(x)$ is increasing on $(-\infty, \infty)$. Now $f''(x) = 24x^2 - 6x + 10$ has no zeroes because the discriminant is $< 0$. Hence $f''(x)$ has the same sign for all $x \in (-\infty, \infty)$. Since $f''(0) = 10$, $f''(x) > 0$ for all $x$. Hence, by the Mean Value Theorem (MVT), $f'(x)$ is an increasing function.

The difference between $f(a + .01)$ and the linear estimate is

\[ f(a + .01) - [f(a) + (.01)f'(a)] = [f(a + .01) - f(a)] - (.01)f'(a) = (.01)f'(c) - (.01)f'(a) \]

for some $c \in (a, a + .01)$ by the MVT. Since $c > a$, it is clear that the value $f(a + .01)$ is greater than the estimate.

Those familiar with the Lagrange form of the remainder for a Taylor series may choose to point out that there exists $c \in (a, a + .01)$ such that

\[ f(a + .01) = f(a) + (.01)f'(a) + \frac{(.01)^2}{2}f''(c). \]