1. (a) Product Rule: \( \frac{d}{dx} (e^x \tan x) = e^x \tan x + e^x \sec^2 x. \)

(b) Chain Rule: \( f'(x) = 2001 \left( x^2 + \sqrt{\frac{x - \pi}{7}} \right)^{2000} \left[ 2x + \frac{1/7}{2\sqrt{\frac{x - \pi}{7}}} \right]. \)

(c) \( g'(t) = \frac{2}{t} \cos(2 \ln t),\ g''(t) = -\frac{4}{t^2} \sin(2 \ln t) - \frac{2}{t^2} \cos(2 \ln t), \) so \( g''(1) = -2. \)

(d) Quotient Rule: \( \frac{du}{dx} = \frac{2x \cos(x^2) \left( 1 + \cos^2 x \right) - \sin(x^2) \left( -2 \cos x \sin x \right)}{(1 + \cos^2 x)^2}. \)

2. Parallel lines have equal slopes, so solve \( \sqrt{1-x^2} = \frac{2}{\sqrt{3}} \) for \( x = \pm \frac{1}{2}. \)

Two points on the curve have the desired property, namely \( \left( \frac{1}{2}, \frac{\pi}{6} \right) \) and \( \left( -\frac{1}{2}, -\frac{\pi}{6} \right). \)

3. By definition,
\[
 f'(3) = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h \to 0} \frac{1}{h} \left[ \frac{1}{(3+h)^2} - \frac{1}{3^2} \right] = \lim_{h \to 0} \frac{-6-h}{(3+h)^2} = \frac{-2}{27}.
\]

4. \( 1 = yy' = (k\sqrt{5x+1}) \left( \frac{5k}{2\sqrt{5x+1}} \right) = \frac{5}{2} k^2. \) Given \( k > 0, \) conclude \( k = \sqrt{\frac{2}{5}}. \)

5. (a) Implicit differentiation gives \( 3^x (\ln 3) - 2^y (\ln 2) y' = 0. \) Plugging in \( (x, y) = (2, 3) \) gives the slope; the tangent line is
\[
 y = 3 + \left( \frac{9 \ln 3}{8 \ln 2} \right) (x - 2).
\]

(b) The original equation implies \( 2^y = 3^x - 1. \) Substitution in (a) gives
\[
 y' = \frac{3^x \ln 3}{(3^x - 1) \ln 2} = \frac{\ln 3}{\ln 2 [1 - 3^{-x}]}.
\]

As \( x \to \infty, \ 3^{-x} \to 0, \) so \( y' \to \frac{\ln 3}{\ln 2}. \)

Another approach is to solve for \( y = \ln(3^x - 1)/\ln 2 \) from the given equation and use direct methods to find either the tangent line in (a) or the limit in (b), or both.
6. Write $y$ for the vertical distance from the bottom of the wall to the top of the ladder. Then the desired area is $A = \frac{1}{2} sy$, and Pythagoras gives $s^2 + y^2 = 5^2$. These observations can be combined before or after differentiating, as follows.

(i) Solving for $y = \sqrt{25 - s^2}$ leads to

$$A = \frac{1}{2} s \sqrt{25 - s^2}, \quad \text{so } \frac{dA}{dt} = \frac{dA}{ds} \frac{ds}{dt} = \frac{1}{2} \left( \sqrt{25 - s^2} - \frac{s^2}{\sqrt{25 - s^2}} \right) (1 + e^{-s}).$$

At the given instant, $y = 3$ and $s = 4$, so $\frac{dA}{dt} = -\frac{7}{6} (1 + e^{-4})$.

(ii) Differentiating $s^2 + y^2 = 25$ gives

$$2s \frac{ds}{dt} + 2y \frac{dy}{dt} = 0, \quad \text{i.e., } \frac{dy}{dt} = -\frac{s}{y} \frac{ds}{dt} = -\frac{s}{y} (1 + e^{-s}).$$

Differentiating $A = \frac{1}{2} sy$ and substituting from the line above yields

$$\frac{dA}{dt} = \frac{1}{2} \left( \frac{ds}{dt} \right) y + \frac{1}{2} s \left( \frac{dy}{dt} \right) = \frac{1}{2} \left( y - \frac{s^2}{y} \right) (1 + e^{-s}),$$

just as before.

Since $\frac{dA}{dt} < 0$, the area is decreasing; its exact rate of change, in $m^2/s$, is $-\frac{7}{6} (1 + e^{-4})$.

7. The differential equation implies that $I(x) = I(0)e^{-kx}$. Measuring $I$ in percent gives $I(0) = 100$ and $I(1) = 60$, so $k = -\ln(3/5)$. The desired thickness $x$, in mm, satisfies

$$1 = I(x) = 100e^{-kx}, \quad \text{i.e., } x = \frac{\ln(100)}{\ln(5/3)} \approx 9.01515.$$  

8. Since $E$ is everywhere differentiable, the desired property is equivalent to the assertion that $E'(t) \leq 0$ always. We prove this by computing $E'$ with the chain rule, then substituting from the given differential equation:

$$\frac{dE}{dt} = 2y(t)y'(t) + 2y'(t)y''(t) = 2y'(t) [y(t) + y''(t)] = 2y'(t) [-cy'(t)] = -2c [y'(t)]^2.$$  

The right side is nonpositive because $c \geq 0$ is given.

9. (a) $v = \frac{ds}{dt} = e^{-t} (\cos t - \sin t) > 0 \iff 0 < t < \frac{\pi}{4}$ (recall $0 < t < \pi$).

(b) $a = \frac{dv}{dt} = -2e^{-t} \cos t > 0 \iff \frac{\pi}{2} < t < \pi$ (recall $0 < t < \pi$).
10. (a) The line through \((2, 2)\) and \((1, 0)\) has slope 2; since it is tangent to the curve \(y = f(x)\) at the point \((2, 2)\), we have \(f'(2) = 2\). (An algebraic solution is also possible: one rearranges Newton’s update formula \(x_1 = x_0 - f(x_0)/f'(x_0)\) to get \(f'(x_0) = 2\).)

(b) Since \(f''(x) < 0\) always, the curve \(y = f(x)\) is concave down. Hence it lies below its tangents, one of which we have just discussed. In particular, \(f(1) < 0\) while \(f(2) > 0\). Existence of \(f''\) implies continuity of \(f\), so the change in sign of \(f\) between \(x = 1\) and \(x = 2\) indicates that it must have a zero between these points.

11. Let \(r\) and \(h\) be the radius and height of the inscribed cylinder. Similar triangles give

\[
\frac{H - h}{r} = \frac{H}{R}
\]

or

\[
\frac{h}{R - r} = \frac{H}{R}.
\]

Only one such equation is needed. Substituting it into the volume formula \(V = \pi r^2 h\) leads to one of

\[
V = \pi R^2 H \left( \frac{r}{R} \right)^2 \left( 1 - \frac{r}{R} \right)
\]

or

\[
V = \pi R^2 H \left( \frac{h}{H} \right)^2 \left( 1 - \frac{h}{H} \right)^2.
\]

We illustrate with the first: since \(g(x) = x^2(1 - x)\) has critical points at \(x = 0\) and \(x = 2/3\), and the first derivative test shows that the choice \(x = 2/3\) gives an absolute maximum for \(g\) over \(0 \leq x \leq 1\), it follows that \(r/R = 2/3\), i.e., \(r = (2/3)R\). The similar-triangles equation above then gives \(h = (1/3)H\).

12. The function described here is decreasing and concave up for \(x < -2\), decreasing and concave down for \(-2 < x < 0\), decreasing and concave up for \(0 < x < 2\), and increasing and concave up for \(x > 2\). A correct graph should show these features, and mention a (horizontal) point of inflection at \((-2, 0)\), another point of inflection at \((0, -2)\), a local and absolute minimum at \((2, y)\) for some unknown \(y < -2\), and an \(x\)-intercept at the point \((4, 0)\).

13. (a) Note that \(f'(x) = 12x^3 + 12(k - 2)x^2 - 6kx\), \(f''(x) = 36x^2 + 24(k - 2)x - 6k\). Since \(f'(0) = 0\) and \(f''(0) = -6k\), the second derivative test implies that \(x = 0\) provides a local maximum for \(f\) when \(k > 0\), and a local minimum (in particular, no local maximum) when \(k < 0\). To settle the borderline case \(k = 0\), write

\[
f(x) = 3x^4 - 8x^3 = x^3(3x - 8) \quad \text{[case } k = 0\text{].}
\]

Here \(f(0) = 0\), but \(f(x) < 0\) for small \(x > 0\) and \(f(x) > 0\) for small \(x < 0\), so there is no local maximum at \(x = 0\) when \(k = 0\). The desired \(k\)-values are exactly those satisfying \(k > 0\).

(b) Assuming \(k > 0\), the local minima are provided by the nonzero roots of

\[
f'(x) = 6x \left[ 2x^2 + 2(k - 2)x - k \right].
\]
By the quadratic formula, these are \(-\frac{1}{2}(k - 2) \pm \frac{1}{2}\sqrt{(k - 2)^2 + 2k}\), and the distance between them is

\[ s = \sqrt{(k - 2)^2 + 2k}. \]

(c) Elementary algebra gives

\[ s = \sqrt{(k^2 - 4k + 4) + 2k} = \sqrt{(k - 1)^2 + 3}. \]

The choice \(k = 1\) clearly minimizes this over all real \(k\); the restriction to \(k > 0\) is required to justify using the form of \(s\) derived in part (b).