Sample Calculus Exam #3

Instructions: Show all your work. Any calculator that is allowed in the Principles of Mathematics 12 provincial examination may be used, but will not be necessary. Answers that are "calculator ready," like $3 + \ln 7$ or $e^{\sqrt{2}}$, are fully acceptable.

1. [9] a) Let $y = x \cos 3x$. Find $\dfrac{dy}{dx}$.

b) Let $y = \left(\dfrac{x+1}{x-1}\right)^{1/2}$. Find $\dfrac{dy}{dx}$.

c) Let $y = \dfrac{1}{1-4x}$. Find $\dfrac{d^2y}{dx^2}$.

2. [5] Find the constant k such that the function $y = 10^t$ satisfies the differential equation $\dfrac{dy}{dt} = ky$.

3. [6] Let $f(x) = \dfrac{5}{3x-1}$. Calculate $f'(2)$ directly from the *definition* of derivative, showing your work in detail.

4. [8] The function y satisfies the equation $4x + 2y = xy^3$. Find the values of $\dfrac{dy}{dx}$ and of $\dfrac{d^2y}{dx^2}$ at the point $(1, 2)$.

5. [5] Evaluate $\displaystyle\lim_{x \to 0} \frac{xe^{3x}}{\tan \pi x}$. Explain your reasoning.

6. [5] Find all values of a for which the tangent line to $y = \arctan x$ at $x = a$ is parallel to the tangent line to $y = \ln(x+2)$ at $x = a$.

7. [7] It takes 27 years for 75% of any quantity of the radioactive isotope Actinium 227 to decay. Let $f(a)$ be the amount of time it takes for 100 milligrams of Actinium 227 to decay to a milligrams. Find a formula for $f(a)$.

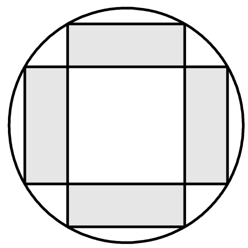
8. [5] The function y satisfies the differential equation

$$\frac{dy}{dx} = \frac{1}{1+3xy}.$$

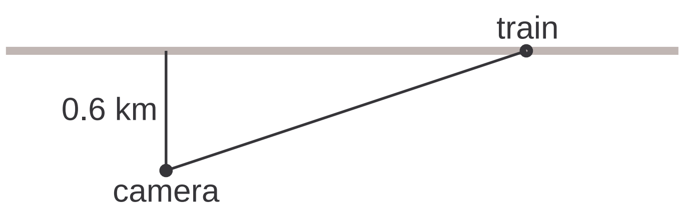
Given that $y(2) = 4$, use a suitable tangent line approximation to estimate $y(2.1)$.

9. [10] A spaceship's thruster produces a constant acceleration of $20\,\text{m/s}^2$. The ship is initially stationary in deep space. The thruster is turned on for 10 seconds. Then the thruster is turned off and the ship coasts for 10 seconds. Finally, the thruster is turned on for another 10 seconds. How far has the ship travelled during the 30 second period just described?

10. [10] Two identical rectangles are inscribed in a circle of radius 1, so that their long sides meet at right angles. Together, the two rectangles form a cross, and the part they have in common is a square. What dimensions of the rectangles maximize the area that lies inside the cross but outside the central square? (This area is shaded in the figure below.)



11. [10] A high speed train is traveling at 2 km/min along a straight track. The train is moving away from a movie camera which is located 0.6 km from the track. The camera keeps turning so as to always point at the front of the train. How fast (in radians per minute) is the camera rotating when the front of the train is 1 km from the camera?



12. [10] A particle is travelling in the plane. At any time t, its x-coordinate is given by $x = \sin^2 t$, and its y-coordinate is given by $y = \cos t$. What is the closest it ever gets to the point $(0, 0)$?

13. [10] A tall cylindrical water tank has a circular base of area 10 square meters. The tank is initially empty. Water is being pumped into the tank through a hose at 0.06 cubic meters per minute. Water is also flowing out though a hole at the bottom of the tank. The outflow rate is variable: at any moment, if the depth of water in the tank is h, then water is flowing out at $0.02h$ cubic meters per minute.

a) If $V(t)$ is the volume of water in the tank at time t, then V satisfies a differential equation of the form $\dfrac{dV}{dt} = c + kV$. What are the numbers c and k?

b) Let $W = c + kV$. Express $\dfrac{dW}{dt}$ in terms of W.

c) Find formulas for W and V as functions of time.