Sample Calculus Exam #2

Instructions: Show all your work. Any calculator that is allowed in the Principles of Mathematics 12 provincial examination may be used, but will not be necessary. Answers that are "calculator ready," like $3 + \ln 7$ or $e^{\sqrt{2}}$, are fully acceptable.

1. [20] a) Let $y = \ln\left(\dfrac{x^2}{e^x}\right)$. Find $\dfrac{dy}{dx}$.

b) Let $f(x) = \sin 3x$. Find $f^{(6)}(x)$, the sixth derivative of $f(x)$.

c) Let $f(x) = 3^{\cos x}$. Find $f'(x)$.

d) Find and simplify $f'(0)$, given

$$f(x) = (x+1)(2x+1)(3x+1)(4x+1)(5x+1).$$

e) An increasing differentiable function y satisfies the equation $\sec y = x$ in the interval $1 \le x < \infty$. Evaluate $\dfrac{dy}{dx}$ at $x = 2$.

2. [5] Let $f(x) = \arctan x + \arctan(1/x)$. Find $f'(x)$ and simplify. Use the result to sketch the curve $y = f(x)$.

3. [5] Find all values of the constant k for which the function $y = e^{kt}$ satisfies the differential equation

$$\frac{d^2y}{dt^2} = \frac{dy}{dt} + y.$$

4. [5] Newton's Method is being used to solve the equation $f(x) = 0$, starting from the initial estimate $x_0 = 4$. The tangent line to $y = f(x)$ when $x = 4$ has equation $y = (5/3)(x-4) - 1/3$. Find the next Newton's Method estimate x_1.

5. [5] Let a be a positive constant. Find the largest value of $(a+x)^3(a-x)^5$ as x ranges over the interval $-a \le x \le a$.

6. [5] Let $f(x) = x^2 e^{-100x}$. Over what interval(s) is $f(x)$ increasing?

7. [5] Find $\displaystyle\lim_{x \to 1} \frac{x^{10} - 1}{x - 1}$.

8. [5] Given that $f(3) = 5$ and that $f'(x) = (x^2 - 1)^{1/3}$, find the tangent line approximation to $f(3.1)$. Is this tangent line approximation larger or smaller than the true value of $f(3.1)$? Explain.

9. [5] Find the constant k for which the line $y = x$ is tangent to the curve $y = k \ln x$.

10. [10] Let $W(t)$ be a baby's weight t days after birth. For the first few weeks of life, $W(t)$ satisfies (approximately) the differential equation $W'(t) = kW(t)$ for some constant k. A baby that weighed $4\,\mathrm{kg}$ at birth weighed $4.84\,\mathrm{kg}$ after 14 days.

a) How much did the baby weigh 7 days after her birth?

b) At what rate was the baby's weight changing 7 days after her birth?

11. [10] A particle is travelling along the curve $y = x^2 - 5$. At the instant that the particle reaches the point $(3, 4)$, the particle's distance from the origin is increasing at the rate of 2 units per second. How fast is the particle's x-coordinate increasing at this instant?

12. [10] A particle is travelling on the x-axis. Denote its x-coordinate at time t by $s(t)$. We know that $s(0) = s'(0) = 0$, and that the acceleration $s''(t)$ of the particle at any time t is given by the formula $s''(t) = \sin 2t$.

a) Find an explicit formula for $s'(t)$.

b) Find the displacement s at $t = \pi/4$.

13. [10] When a fish swims upstream for t hours at speed v, *relative to the water*, the total amount of energy it expends is approximately equal to kv^3t for some constant k. Assume that salmon swim at constant speed, in a way that minimizes the total energy expended. How long will it take a salmon to go 240 kilometers upstream against a steady 6 kilometer per hour current?