Calculus Challenge Exam

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June 15, 2022, 12:00-15:00 PDT

Name: ____________________

Signature: ____________________

School: ____________________

Candidate number: ____________________

Rules and instructions:

1. Show all your work! Full marks are given only when the answer is correct, and is supported with a written derivation that is orderly, logical, and complete. Part marks are available in every question.
2. Calculators are optional, not required. Correct answers that are "calculator ready," like $3+\ln(7)$ or e^2, are fully acceptable.
3. Only dedicated calculators with cleared memories and no graphing capabilities may be used.
4. A formula sheet will be provided to you. No other notes, books, or aids are allowed. In particular, all calculator memories must be empty when the exam begins.
5. *Candidates guilty of any of the following or similar practices shall be dismissed from the examination immediately and assigned a grade of 0:*
   (a) Using any books, papers or memoranda.
   (b) Speaking or communicating with other candidates.
   (c) Exposing written papers to the view of other candidates.
6. Do not write in the grade box shown to the right.

For examiners' use only		
Question	Points	Score
1	8	
2	8	
3	8	
4	8	
5	8	
6	6	
7	6	
8	15	
9	18	
10	15	
Total	100	

1. **[8 marks]** Each part is worth 4 marks. Write your final answer in the box. No credit will be given for answers without accompanying work.

(a) Evaluate $\lim_{h \to 0} \frac{(8+h)^{-1} - 8^{-1}}{h}$.

Answer:

(b) Evaluate $\lim_{x \to -\infty} \frac{\sqrt{2+10x^2}}{4+8x}$.

Answer:

2. [**8 marks**] Each part is worth 4 marks. Write your final answer in the box. No credit will be given for answers without accompanying work.

(a) Find the slope of the tangent line to $y = \frac{\sqrt{x}}{x+5}$ at the point $\left(4, \frac{2}{9}\right)$.

Answer:

(b) Find the equation of the tangent line to $y = 5\sec(x) - 10\cos(x)$ at the point $(0, -5)$. Your answer should be in the form $y = mx + b$.

Answer:

3. [**8 marks**] Each part is worth 4 marks. Write your final answer in the box. No credit will be given for answers without accompanying work.

(a) Let $f(x) = x + 2\sin(x)$. Find all values of x in the interval $[0, 2\pi]$ for which the graph has a horizontal tangent line.

Answer:

(b) Suppose the derivative of $f\left(x^2\right)$ with respect to x is equal to x^6. What is $f'\left(x^2\right)$?

Answer:

4. **[8 marks]** Each part is worth 4 marks. Write your final answer in the box. No credit will be given for answers without accompanying work.

(a) Find $\frac{dy}{dx}$ given that $\sqrt{x+y} = 7 + x^2y^2$.

Answer:

(b) Find $\frac{dy}{dx}$ given that $e^{x^2y} = x + y$.

Answer:

5. [**8 marks**] Each part is worth 4 marks. Write your final answer in the box. No credit will be given for answers without accompanying work.

(a) Let $f(x) = x^4 - 6x^3$. At what x-value does $f(x)$ attain its minimum?

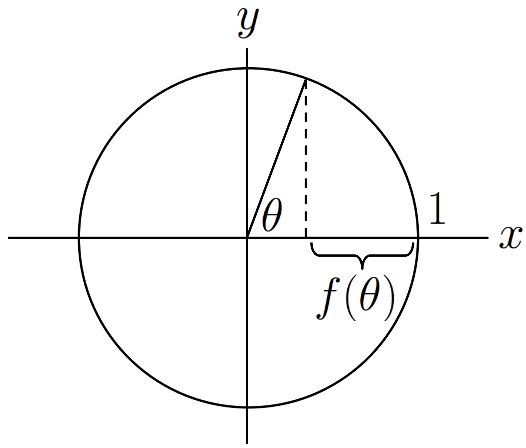
Answer:

(b) Find the maximal area of a right triangle with hypotenuse of length 1.

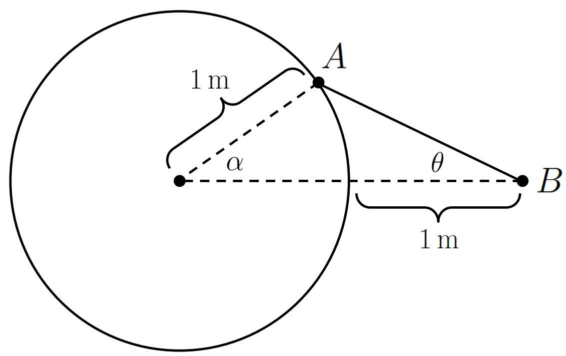
Answer:

6. [**6 marks**] Come up with a differentiable function $f(x)$ that has exactly two extrema: an absolute or global minimum at $x = -3$, and an absolute or global maximum at $x = -2$.

7. [**6 marks**] Calculate the derivative of the function $f(\theta)$ illustrated below.



8. [**15 marks**] Imagine a disc of radius 1 m spinning anticlockwise at π rad/sec. A tight elastic connector joins the point A on the rim of the disc to the stationary point B located 1 m away from the disc, as shown below.



Find the rate of change of the angle θ at the moment $\alpha = \frac{\pi}{3}$. (Hint: begin by writing $\tan(\theta)$ in terms of α.)

(You may use this page to continue your answer to question 8.)

9. **[18 marks]** Consider the curve $x^3 + y^3 = 1$.

(a) Find its intercepts. Write your final answer in the box, with intercepts expressed in the form (x_0, y_0). No credit will be given for an answer without accompanying work.

Answer:

(b) The curve has an *oblique* or *slant* asymptote — that is, a line whose vertical distance from the curve is arbitrarily small provided x is sufficiently large. What is the equation of the oblique asymptote? Justify your answer below, and write your final answer in the box, with the asymptote expressed in the form $y = mx + b$. No credit will be given for an answer without justification.

Answer:

(c) The first derivative of y with respect to x, $\frac{dy}{dx}$, is equal to $-\frac{x^2}{y^2}$. Using this fact, determine at what point(s) the curve has a horizontal tangent line. Write your final answer in the box, with point(s) expressed in the form (x_0, y_0). No credit will be given for an answer without accompanying work.

Answer:

(d) Determine at what point(s) the curve has a vertical tangent line. Write your final answer in the box, with point(s) expressed in the form (x_0, y_0). No credit will be given for an answer without accompanying work.

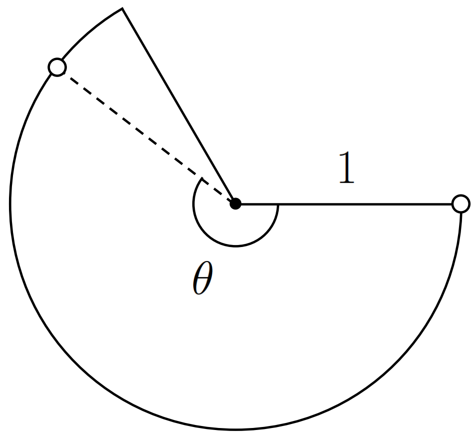
Answer:

(e) Using the first derivative given in part (c), the second derivative may be calculated and written in the form $\frac{f(x)}{y^5}$ where $f(x)$ is a function that depends only on x (and not on y). Find $f(x)$. Write your final answer in the box. No credit will be given for an answer without accompanying work.

Answer:

(f) Using the information determined in the previous parts of this question, sketch a large graph of the curve below, indicating, if you have determined them, its intercepts, oblique asymptote, and where it has horizontal or vertical tangent lines, *as well as where it is concave up and where it is concave down.*

10. **[15 marks]** We wish to construct a cone using a segment of a circle of paper of radius 1, as shown below. The cone is constructed by cutting from the open dot straight to the centre of the circle, and then to the other open dot, and then gluing together the cut edges.



Find the angle θ that gives us a cone of maximal volume, under the restriction that $0 \leq \theta \leq \frac{4\pi}{3}$.

(You may use this page to continue your answer to question 10.)