

Calculus Challenge Exam

UBC • SFU • TWU • UFV • UNBC • UVic

June 6, 2019, 12:00-15:00 PDT

Name: _____

Signature: _____

School: _____

Candidate number: _____

Rules and instructions:

1. Show all your work! Full marks are given only when the answer is correct, and is supported with a written derivation that is orderly, logical, and complete. Part marks are available in every question.
2. Calculators are optional, not required. Correct answers that are “calculator ready,” like $3+\ln(7)$ or e^2 , are fully acceptable.
3. Only dedicated calculators with cleared memories and no graphing capabilities may be used.
4. A formula sheet will be provided to you. No other notes, books, or aids are allowed. In particular, all calculator memories must be empty when the exam begins.
5. If you need more space to solve a problem on page n , work on the back of page $n - 1$.
6. *Candidates guilty of any of the following or similar practices shall be dismissed from the examination immediately and assigned a grade of 0:*
 - (a) Using any books, papers or memoranda.
 - (b) Speaking or communicating with other candidates.
 - (c) Exposing written papers to the view of other candidates.
7. Do not write in the grade box shown to the right.

For examiners' use only		
Question	Points	Score
1	48	
2	8	
3	8	
4	16	
5	12	
6	8	
Total	100	

1. [48 marks] Each part is worth 4 marks. Write your final answer in the box. No credit will be given for answers without accompanying work.

(a) Determine where the function $f(x) = \frac{\ln(x-2)}{\sin\left(\frac{x}{2}\right)}$ is *not* defined.

Answer:

(b) Evaluate the limit $\lim_{x \rightarrow -\infty} \frac{1+x+x^3}{\sqrt{2+3x^3+4x^6}}$.

Answer:

(c) Evaluate the limit $\lim_{x \rightarrow 0} \frac{5x \cos(2x)}{4 \sin(3x)}$.

Answer:

(d) Evaluate the limit $\lim_{x \rightarrow -\infty} x e^x \cos(x)$.

Answer:

- (e) Give an example of a continuous function with one horizontal asymptote at $y = -1$ and one horizontal asymptote at $y = e$.

Answer:

- (f) Determine the number of times the curve $y - \log(x) = 2^{\sqrt{x}}$ crosses the x -axis.

Answer:

(g) Let $f(x)$ satisfy $f(2) = f'(2) = 2$, and $g(x) = \frac{f(f(x))}{f(x)}$. Calculate $g'(2)$.

Answer:

(h) Find the slope of the line tangent to the curve $y = x^x$ at $x = e$.

Answer:

(i) Find the slope of the line tangent to the curve $x^4 - x^2y + y^4 = 1$ at the point $(1, 1)$.

Answer:

(j) Find an expression for $L(x)$, the linear approximation of $f(x) = \frac{x}{1 + \frac{1}{x}}$ at $x = 1$.

Answer:

- (k) Give two examples of a function $f(x)$ satisfying $\frac{f'(x)}{f(x)} = 5$ for all x .

Answer:

- (l) Give an example of a polynomial function that has exactly three local extrema, at $x = 0$, $x = 1$ and $x = -1$.

Answer:

In the remaining problems, show all your work unless stated otherwise. Write your final answer in the box where there is a box provided. No credit will be given for answers without accompanying work.

2. [8 marks] Let $f(x)$ be differentiable.

(a) State the limit definition of $f'(x)$. (You do not have to show your work.)

Answer:

(b) Use the limit definition of derivative to find the derivative of $f(x)^2$, making sure to justify each step. Your answer may be in terms of $f(x)$ and $f'(x)$.

3. [8 marks] Let $f(x) = x^3 - 3x + 5$.

(a) Find the equation(s) of all horizontal tangent lines to the curve $y = f(x)$.

Answer:

(b) Determine all values m such that there exists at least one tangent line to the curve $y = f(x)$ of slope m .

Answer:

4. [16 marks] Let $f(x) = \frac{\ln x}{x}$.

(a) State the domain of $f(x)$. (You do not have to show your work.)

Answer:

(b) State the intercepts of $f(x)$. (You do not have to show your work.)

Answer:

(c) Determine if $f(x)$ has any horizontal asymptotes.

Answer:

(d) Determine if $f(x)$ has any vertical asymptotes.

Answer:

(e) Calculate $f'(x)$.

Answer:

(f) Determine the intervals where $f(x)$ is increasing, and the intervals where $f(x)$ is decreasing.

Answer:

(g) Calculate $f''(x)$.

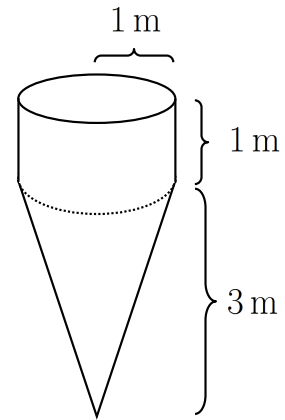
Answer:

(h) Determine the intervals where $f(x)$ is concave up, and the intervals where $f(x)$ is concave down.

Answer:

- (i) Draw a large sketch of the graph of $f(x)$ below, making sure to include all the features determined in the previous parts of the question.

5. [12 marks] Consider a tank full of water whose top part is a cylinder of radius 1 m and height 1 m, and whose bottom part is a cone of radius 1 m and height 3 m, as shown. Now suppose the tank is drained at constant rate of $1 \text{ m}^3/\text{min}$.



- (a) Find the rate at which the water level is dropping 2 minutes after the tank begins to drain.

Answer:

- (b) Find the rate at which the water level is dropping 4 minutes after the tank begins to drain.

Answer:

6. [8 marks] Find the area of the largest rectangle that can be inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Answer: