SFU-UBC-UNBC-UVic

CALCULUS CHALLENGE EXAMINATION

JUNE 07, 2018, 12:30 - 15:30

Name: ______________________________ (please print)
family name *given name*

Signature: ______________________________

Instructions:

1. Do not open this booklet until told to do so.
2. Write your name above in block letters and sign your exam.
3. Write your answer in the space provided below the question. If additional space is needed then use the back of the previous page. Your final answer should be simplified as far as is reasonable.
4. Make the method you are using clear in every case unless it is explicitly stated that no explanation is needed.
5. This exam has 15 questions on 15 pages (not including this cover page and the sheet of trigonometric identities). Once the exam begins please check to make sure your exam is complete.
6. **No** Books, papers, or electronic devices except a Ministry-approved calculator and the sheet of trigonometric identities attached to this examination shall be within the reach of a student during the examination.
7. **During the examination, communicating with, or deliberately exposing written papers to the view of, other examinees is forbidden.**

Question	Maximum	Score
1	10	
2	10	
3	8	
4	8	
5	8	
6	6	
7	6	
8	12	
9	8	
10	6	
11	8	
12	12	
13	10	
14	12	
15	8	
Total	132	

1. Compute the following limits.

[3] (a) $\lim_{x\to 1} \dfrac{x^2-9}{x^2+2x-3}$ does not exist since

$\lim_{x\to 1} (x^2+2x-3) = 0$ whereas $\lim_{x\to 1} (x^2-9) \neq 0$

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[3] (b) $\lim_{x\to 8^-} \dfrac{8-x}{|x-8|}$ $= \lim_{x\to 8^-} \dfrac{8-x}{-(x-8)}$

$= \lim_{x\to 8^-} 1$

$= 1$

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[4] (c) $\lim_{t\to 0} \dfrac{\sqrt{1+t}-\sqrt{1-t}}{t} \cdot \dfrac{\sqrt{1+t}+\sqrt{1-t}}{\sqrt{1+t}+\sqrt{1-t}}$

$= \lim_{t\to 0} \dfrac{(1+t)-(1-t)}{t(\sqrt{1+t}+\sqrt{1-t})}$

$= \lim_{t\to 0} \dfrac{2t}{t(\sqrt{1+t}+\sqrt{1-t})}$

$= \lim_{t\to 0} \dfrac{2}{\sqrt{1+t}+\sqrt{1-t}}$

$= 1$

General Comments

1(a): Students were mostly able to recognize that the limit does not exist but incorrectly wrote the answer as infinity or -infinity.

1(b): Most students evaluting the limit correctly and were able to deal with the absolute value. Some students investigated the limit numerically rather than using properties of limits and did not receive much credit for this.

1(c): Most students found the strategy of multiplying the numerator and denominator by the conjugate but some had difficulty simplifying the result.

2. Calculate dy/dx.

[3] (a) $y = 2e^x \cdot \tan\left(\frac{1}{x}\right)$

$$\frac{dy}{dx} = \left[\frac{d}{dx}(2e^x)\right] \cdot \tan\left(\frac{1}{x}\right) + 2e^x \cdot \left[\frac{d}{dx}\tan\left(\frac{1}{x}\right)\right]$$

$$= 2e^x \cdot \tan\left(\frac{1}{x}\right) + 2e^x \cdot \sec^2\left(\frac{1}{x}\right) \cdot \left(-\frac{1}{x^2}\right)$$

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[3] (b) $y = \frac{\ln(\cos x)}{\sqrt[3]{x}}$

$$\frac{dy}{dx} = \frac{\sqrt[3]{x} \cdot \left[\frac{d}{dx}\ln(\cos x)\right] - \left[\frac{d}{dx}\sqrt[3]{x}\right] \cdot \ln(\cos x)}{(\sqrt[3]{x})^2}$$

$$= \frac{-\sqrt[3]{x}\tan x - \frac{1}{3}x^{-2/3}\ln(\cos x)}{x^{2/3}}$$

$$= -x^{-1/3}\tan x - \frac{1}{3}x^{-4/3}\ln(\cos x)$$

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[4] (c) $xe^y = y\sin x$

$$\frac{d}{dx}(xe^y) = \frac{d}{dx}(y\sin x)$$

$$e^y + xe^y \cdot \frac{dy}{dx} = \frac{dy}{dx} \cdot \sin x + y\cos x$$

$$\frac{dy}{dx} = \frac{y\cos x - e^y}{xe^y - \sin x}$$

General Comments

Overall this question was very well done. Common mistakes include:

2(a): Improper usage of the product rule: $[uv]' = u'v'$

2(b): Wrong quotient rule: $[u/v]' = (u'v + uv')/v^2$

2(c): Trying to solve explicitly for y
Incorrectly writing $xe^y = (xe)^y$
Incorrect properties of logs: $\ln(ab) = \ln(a)\ln(b)$

[8] **3.** Let $f(x) = \sqrt{3-x}$. Use the definition of the derivative (not differentiation rules) to find the derivative of f at $x = -1$.

$$f'(1) = \lim_{h \to 0} \frac{f(-1+h) - f(-1)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{4-h} - \sqrt{4}}{h}$$

$$= \lim_{h \to 0} \frac{(4-h) - 4}{h(\sqrt{4-h} + \sqrt{4})}$$

$$= \lim_{h \to 0} \frac{-1}{\sqrt{4-h} + 2}$$

$$= -\frac{1}{4}$$

General Comments

3: The majority of students solved this problem, though many could not recall the definition of the derivative correctly. A common mistake was to write f(x + h) = sqrt(3 - x + h) for f(x) = sqrt(3 - x).

[8] **4.** Find the curve $y = f(x)$ where $\frac{d^2y}{dx^2} = 6x$ and the graph of f passes through the point $(1, 2)$ with a horizontal tangent there.

This is an initial value problem with

$$f''(x) = 6x, \quad f(1) = 2, \quad f'(1) = 0.$$

Integrating yields

$$f'(x) = 3x^2 + C_1$$

and the initial condition $f'(1) = 0$ gives $C_1 = -3$.

Thus

$$f'(x) = 3x^2 - 3.$$

Integrating again we obtain

$$f(x) = x^3 - 3x + C_2$$

and the initial condition $f(1) = 2$ gives $C_2 = 4$.

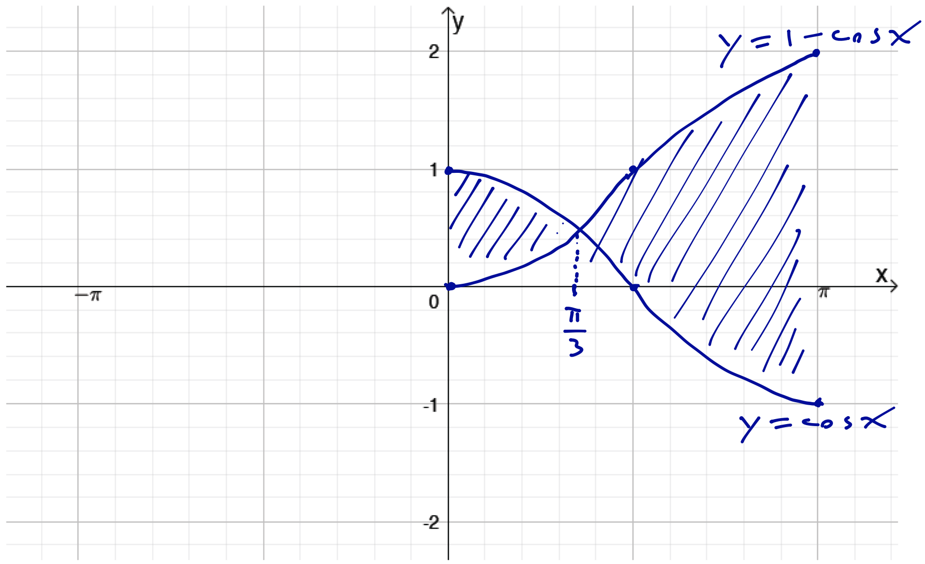
Therefore the equation of the curve is

$$f(x) = x^3 - 3x + 4.$$

General Comments

4: Overall this question was well done. Students that found the general form of the function were able to solve for the constants correctly. Some students forgot to include one or more contants of integration.

[8] **5.** Sketch the region enclosed between the curves $y = \cos(x)$ and $y = 1 - \cos(x)$ on $[0, \pi]$ in the space provided below and find its area.



Intersection point: $\cos x = 1 - \cos x$

$\cos x = \frac{1}{2}$

$x = \pi/3$

$$\text{Area} = \int_0^{\pi/3} \left[\cos x - (1 - \cos x)\right] dx + \int_{\pi/3}^{\pi} \left[(1 - \cos x) - \cos x\right] dx$$

$$= \int_0^{\pi/3} (2\cos x - 1)\, dx + \int_{\pi/3}^{\pi} (1 - 2\cos x)\, dx$$

$$= \left[2\sin x - x\right]_0^{\pi/3} + \left[x - 2\sin x\right]_{\pi/3}^{\pi}$$

$$= 2\left(\frac{\sqrt{3}}{2} - 0\right) - \left(\frac{\pi}{3} - 0\right) + \left(\pi - \frac{\pi}{3}\right) - 2\left(0 - \frac{\sqrt{3}}{2}\right)$$

$$= 2\sqrt{3} + \frac{\pi}{3}$$

General Comments

5: This question was well done. The most common error was in identifying the correct region. Some students found the area between the curves only on [0, pi/3]. Other students found the area between the curves on [-pi, pi]. There were also various computation errors in computing the definite integrals.

[6] **6.** Determine the value(s) of a so that g is continuous at $x = 2$. Justify your answer using the **definition of continuity**.

$$g(x) = \begin{cases} ax^2 + 1, & \text{if } x < 2 \\ ax - 5, & \text{if } x \geq 2 \end{cases}$$

We require that $\lim_{x \to 2} g(x) = g(2) = 2a - 5$.

Taking one-sided limits,

$$\lim_{x \to 2^-} g(x) = \lim_{x \to 2^-} (ax^2 + 1) = 4a + 1$$

$$\lim_{x \to 2^+} g(x) = \lim_{x \to 2^+} (ax - 5) = 2a - 5$$

Therefore g will be continuous at $x = 2$ when

$$4a + 1 = 2a - 5$$

$$2a = -6$$

$$a = -3$$

General Comments

6: Many students had difficulty presenting a complete solution that demonstrated their understanding of the definition of continuity at a point. Often there was no mention of limits being taken or it was not made explicit that these limits were compared to the value of the function at $x = 2$. The average score was about 3/6 for this question.

[6] **7.** In this question we investigate the positive solution of the equation

$$x^2 + x = 5 - \ln(x).$$

Starting with an initial approximation for the solution of $x_1 = 1$, what is the value x_2 given by one iteration of Newton's method?

Rewriting the equation as

$$x^2 + x - 5 + \ln x = 0$$

we define $f(x) = x^2 + x - 5 + \ln x$

and apply Newton's method to find the zero.

Newton's iteration formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n^2 + x_n - 5 + \ln x_n}{2x_n + 1 + \frac{1}{x_n}}$$

and so

$$x_2 = 1 - \frac{(1)^2 + 1 - 5 + \ln(1)}{2(1) + 1 + \frac{1}{1}}$$

$$= 1 - \frac{-3}{4}$$

$$= 1.75$$

General Comments

7: This question was done fairly well. The majority of students were able to setup the correction iteration formula but some had difficulty applying it.

8. Suppose that $f(x) = 12x^{1/3} + 3x^{4/3}$ with first and second deriviatives

$$f'(x) = \frac{4(x+1)}{x^{2/3}} \quad \text{and} \quad f''(x) = \frac{4(x-2)}{3x^{5/3}}.$$

[2] (a) Using the derivatives above, find the critical number(s) of f.

Critical numbers are $x = -1$ and $x = 0$.

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[4] (b) Fill in the blanks below. Also, classify the critical number(s) as corresponding to either local maxima, local minima, or neither.

The function f is increasing on the interval(s) $(-1, 0)$ and $(0, \infty)$

The function f is decreasing on the interval(s) $(-\infty, -1)$

local minimum at $x = -1$

neither at $x = 0$

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[4] (c) Fill in the blanks below. Also, identify any inflection points by giving their coordinates.

The function f is concave up on the interval(s) $(-\infty, 0)$ and $(2, \infty)$

The function f is concave down on the interval(s) $(0, 2)$

Inflection points at $(0, 0)$ and $(2, 18\sqrt[3]{2})$

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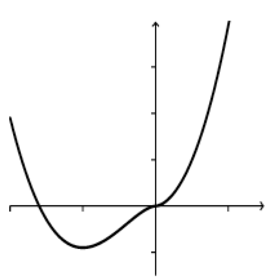
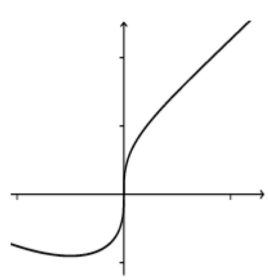
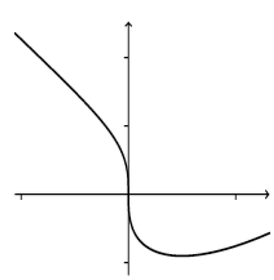
[2] (d) Select the graph of $y = f(x)$ from the curves below by ticking the appropriate box.



A ☐



B ☑ ✓



C ☐

General Comments

8: This question was fairly well done. Many students had difficulty identifying the critical points. The most common error was to overlook the critical point at $x = 0$ where the first derivative is undefined. Some students made the erroneous assumption that the zero of the second derivative was also a critical point.

Many students did not read the question carefully and only gave the x-coordinate for the inflection point.

Almost all students chose the correct graph, though some chose A instead.

[6] **9.** (a) Find the linear approximation of the function $f(x) = \sqrt{x}$ at $a = 4$, and use it to estimate the value of $\sqrt{5}$.

The linear approximation is

$$f(x) \approx f(a) + f'(a)(x-a)$$
$$= \sqrt{a} + \frac{1}{2\sqrt{a}}(x-a)$$
$$= 2 + \frac{1}{4}(x-4)$$
$$= 1 + \frac{1}{4}x$$

Thus $\sqrt{5} = f(5) \approx 1 + \frac{5}{4} = \frac{9}{4}$

[2] (b) Is the approximation that you found in part (a) an underestimate or an overestimate of the actual value? Explain.

Checking the concavity we find that

$$f'(x) = \frac{1}{2\sqrt{x}}$$
$$f''(x) = -\frac{1}{4}x^{-1/2}$$
$$f''(4) < 0$$

Since the curve $y = \sqrt{x}$ lies below its tangent line at $x = 4$, the linear approximation gives an overestimate.

General Comments

9(a): This question was very well done overall. The most common errors were using x = sqrt(5) instead of x = 5 in the approximation formula or writing f(a) - f'(a)(x - a) as an approximation for f(x).

9(b): This question was well done. Most students recognized that they could use the concavity of the graph to determine whether or not the estimate was larger or smaller than the actual value. The most common error was checking the concavity at x = sqrt(5) instead of at x = 4, the point of tangency.

10. A ball is thrown vertically upwards from a platform 12 feet above the ground so that after t seconds have elapsed the height s (in feet) of the ball above the ground is given by

$$s(t) = 12 + 96t - t^2.$$

Compute the following quantities.

[2] (a) The initial velocity.

Velocity : $v(t) = s'(t) = 96 - 2t$

Initial velocity : $v(0) = 96$ ft/s

[2] (b) The time to the highest point.

Setting $v(t) = 0$ yields $t = 48$ s

This is the time to the highest point.

[2] (c) The maximum height attained.

The maximum height attained is

$s(48) = 12 + 96(48) - (48)^2$

$= 2316$ ft

General Comments

10: Overall this question was very well done with no common errors to report.

[8] **11.** Find an equation of the tangent line at $(0, 1)$ to the curve given by the implicitly defined function

$$y \cos x = y^2 + x^3.$$

Differentiating implicitly,

$$\frac{d}{dx}(y \cos x) = \frac{d}{dx}(y^2 + x^3)$$

$$\frac{dy}{dx} \cdot \cos x - y \sin x = 2y \cdot \frac{dy}{dx} + 3x^2$$

$$\frac{dy}{dx} = \frac{3x^2 + y \sin x}{\cos x - 2y}$$

The slope is $\left.\frac{dy}{dx}\right|_{(x,y)=(0,1)} = 0$

Thus the equation of the tangent line is $y = 1$.

General Comments

11: Students had some difficulty with this problem. The most common problems were in differentiating implicitly with respect to x, especially in applying the chain rule to finding d/dx y^2. A few students were confused how to proceed after finding the slope.

[12] **12.** When a cold drink is taken from a refrigerator, its temperature is 5°C. After 25 minutes in a 20°C room its temperature has increased to 10°C. Find a function that models the temperature of the drink t minutes after removing it from the refrigerator and use it to calculate the temperature of the drink after 50 minutes.

Let T denote the temperature (°C) of the drink at time t and let T_s denote the surrounding temperature.

Newton's Law of Cooling gives

$$\frac{dT}{dt} = k(T - T_s)$$

where k is a constant.

Let $y = T - T_s$. Then $\frac{dy}{dt} = \frac{dT}{dt}$ and thus

$$\frac{dy}{dt} = ky$$

whose solution is $y = y(0) \cdot e^{kt}$

or $T - T_s = (T(0) - T_s)e^{kt}$

$$T - 20 = -15e^{kt} \quad \ldots \text{①}$$

At $t = 25$ we have $T = 10$ and so

$$-10 = -15e^{25k}$$

$$k = \frac{\ln 2/3}{25}$$

Substituting into Equation ① yields

$$T = 20 - 15e^{\frac{\ln 2/3}{25}t}$$

$$= 20 - 15\left(\frac{2}{3}\right)^{t/25}$$

After 50 minutes the temperature is

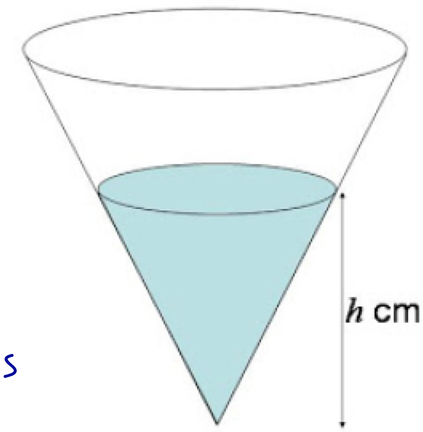
$$T(50) = 20 - 15\left(\frac{2}{3}\right)^2 = \frac{40}{3} = 13.3^\circ C$$

General Comments

12: The vast majority of students did this problem very well, with the most common mistakes coming from those who started from the solution form $T - T_s = (T_0 - T_s)e^{kt}$ without the differential equation. In such cases it was difficult to award partial credit if the solution form was incorrect.

In one particular case, a student did not use Newton's Law of Cooling $dT/dt = k(T - T_s)$ but instead solve a logistic growth problem. This resulted in a qualitatively similar and more sophisticated model. Full credit was given for this solution.

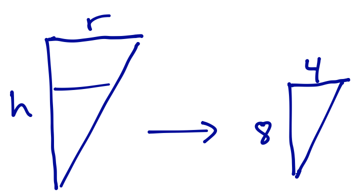
[10] **13.** A coffee filter has the shape of an inverted cone. Suppose coffee drains out the bottom of the cone at a rate of $10\,\text{cm}^3/\text{sec}$. How fast is the height of the liquid changing when the height is 8 cm and the circular surface of the liquid has radius 4 cm. (The volume of a cone is $\frac{1}{3}\pi r^2 h$.)



The height, radius, and volume of the liquid are related by

$$V = \frac{1}{3}\pi r^2 h.$$

We can use similar triangles to eliminate r:



$$r = \frac{1}{2}h$$

Thus the volume is

$$V = \frac{1}{12}\pi h^3$$

and so

$$\frac{dV}{dt} = \frac{1}{4}\pi h^2 \cdot \frac{dh}{dt}$$

When the height is 8 cm we have

$$-10 = \frac{1}{4}\pi (8)^2 \frac{dh}{dt}$$

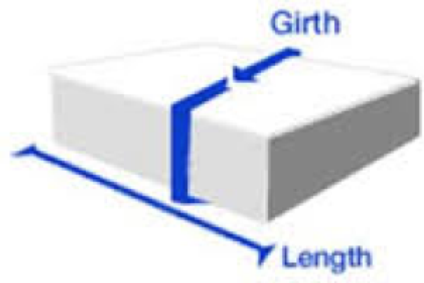
$$\frac{dh}{dt} = -\frac{5}{8\pi}$$

and so the height is decreasing at $\frac{5}{8\pi}$ cm/sec.

General Comments

13: Overall students had difficulty with this question. A common mistake was to take into account that the rate at which the volume was changing is negative. The most common difficulty was finding a correct relation between the height and the radius in order to express the volume as a function of one variable.

[12] **14.** The post office will accept a box for local shipment only if the sum of its length and girth (distance around the box) does not exceed 108 inches. What dimensions will give a box with a square base the largest possible volume?



Let x denote the length and width in inches.

Let y denote the height in inches.

Then the volume is $V = x^2 y$.

A box with maximum volume is subject to the constraint

$$3x + 2y = 108$$
$$y = 54 - \frac{3}{2}x$$

Thus, the function to maximize is

$$V(x) = 54x^2 - \frac{3}{2}x^3 \quad , \quad 0 < x < 36$$

Critical numbers:
$$V'(x) = 108x - \frac{9}{2}x^2$$
$$0 = 108x - \frac{9}{2}x^2$$
$$x = 24$$

Checking that $x = 24$ corresponds to a maximum:

$$V''(x) = 108 - 9x$$
$$V''(24) < 0$$

The corresponding value for y is $y = 54 - \frac{3}{2}(24) = 18$.

Therefore the volume is maximized when the length and width are 24 inches and the height is 18 inches.

General Comments

14: Students struggled with this problem, with the majority of mistakes surrounding correctly setting up the constraint function girth + length = 108. Another common mistake was neglecting to confirm that the critical value corresponds to a maximum.

[8] **15.** Find a function f such that $f'(x) = x^3$ and the line $x + y = 0$ is tangent to the graph of f.

Since f is an antiderivative of f' we have

$$f(x) = \frac{1}{4}x^4 + C \quad \text{for some constant } C.$$

Let $x = a$ correspond to the point of tangency.

Since the tangent line has slope -1, we have

$$f'(a) = -1$$
$$a^3 = -1$$
$$a = -1$$

From the equation of the tangent line, we find that the point of tangency is $(-1, 1)$.

Since this point also lies on the graph of f, we have

$$f(-1) = \frac{1}{4}(-1)^4 + C = 1$$
$$C = \frac{3}{4}.$$

Therefore, the function is

$$f(x) = \frac{1}{4}x^4 + \frac{3}{4}.$$

General Comments

15: This was a fairly nonstandard problem, though many students managed to solve it. The average score was about 6/8. The most common difficulty was in identifying the that (-1, 1) is the point of tangency and therefore a point on both the graph of f and its tangent line there.

Formula Sheet

Exact Values of Trigonometric Functions

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$	π
$\sin\theta$	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0
$\cos\theta$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0	$-1/2$	$-\sqrt{2}/2$	$-\sqrt{3}/2$	-1

Trigonometric Definitions and Identities

$$\sin(-\theta) = -\sin\theta$$

$$\cos(-\theta) = \cos\theta$$

$$\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \sin\phi\cos\theta$$

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

$$\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$

$$\sin^2\theta = \frac{1-\cos(2\theta)}{2}$$

$$\cos^2\theta = \frac{1+\cos(2\theta)}{2}$$

$$\sin^2\theta + \cos^2\theta = 1$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\sec\theta = \frac{1}{\cos\theta}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\csc\theta = \frac{1}{\sin\theta}$$