Calculus Challenge Exam
UBC • SFU • UVic • UNBC

June 8, 2017, 12:00-15:00 PDT

Name: ______________________________

Signature: ______________________________

School: ______________________________

Candidate number: ______________________________

Rules and instructions:

1. Show all your work! Full marks are given only when the answer is correct, and is supported with a written derivation that is orderly, logical, and complete. Part marks are available in every question.
2. Calculators are optional, not required. Correct answers that are "calculator ready," like $3+\ln(7)$ or e^2, are fully acceptable.
3. Only dedicated calculators with cleared memories and no graphing capabilities may be used.
4. A formula sheet will be provided to you. No other notes, books, or aids are allowed. In particular, all calculator memories must be empty when the exam begins.
5. If you need more space to solve a problem on page n, work on the back of page $n-1$.
6. *Candidates guilty of any of the following or similar practices shall be dismissed from the examination immediately and assigned a grade of 0:*
   (a) Using any books, papers or memoranda.
   (b) Speaking or communicating with other candidates.
   (c) Exposing written papers to the view of other candidates.
7. Do not write in the grade box shown to the right.

For examiners' use only		
Question	Points	Score
1	39	
2	8	
3	10	
4	15	
5	10	
6	10	
7	8	
Total	100	

1. **[39 marks]** Each part is worth 3 marks. Write your final answer in the box where there is a box provided. No credit will be given for answers without accompanying work.

   (a) Determine where the function $f(x) = \ln(3 - \ln(x))$ is defined.

   Answer:

   (b) Evaluate the limit $\lim_{x \to 0} \dfrac{x}{\sqrt{x+4} - 2}$.

   Answer:

(c) Evaluate the limit $\lim_{x \to \infty} \dfrac{2x^{5/2}\sqrt{7x+1}}{x + 3x^2 - 5x^3}$.

Answer:

(d) Evaluate the limit $\lim_{x \to 0} \sqrt{x} \sin\left(\frac{1}{x}\right)$.

Answer:

(e) Give an example of a function with exactly one vertical asymptote, at $x = -3$; and exactly one horizontal asymptote, at $y = \pi$.

Answer:

(f) Explain why the curve $y + 2^x = \cos(x^2)$ crosses the x-axis at least once.

(g) Let $f(x)$ be a differentiable function with three roots. Explain why $f'(x)$ has at least two roots.

(h) Let $f(x)$ be a function satisfying $f'\left(\sqrt{2}\right) = \sqrt{3}$, and $g(x) = f(2\sin(x))$. Calculate $g'\left(\frac{\pi}{4}\right)$. Simplify your answer completely.

Answer:

(i) Find the slope of the line tangent to the curve $x^4 - x^2y + y^4 = 1$ at the point $(1, 1)$.

Answer:

(j) Find an expression for $L(x)$, the linear approximation of $f(x) = \frac{3x^2}{x^2-1}$ at $x = 2$.

Answer:

(k) Find the derivative of $f(x) = x^x$.

Answer:

(l) Give an example of a function $f(x)$ satisfying $2f'(x) = 3f(x)$ and $f(0) = 4$.

Answer:

(m) Give an example of a function that has exactly two local extrema, at $x = 2$ and $x = 3$.

Answer:

In the remaining problems, show all your work unless stated otherwise. Write your final answer in the box where there is a box provided. No credit will be given for answers without accompanying work.

2. [**8 marks**] Let $g(x)$ be differentiable and nonzero everywhere.

   (a) State the limit definition of $g'(x)$. (You do not have to show your work.)

Answer:

   (b) Use the limit definition of derivative to find the derivative of $\frac{1}{g(x)}$.

3. [**10 marks**] Let $f(x) = x^2 + x$.

(a) Find the equation of the line tangent to the curve $y = f(x)$ at $(-1, 0)$.

Answer:

(b) Find both points on the curve $y = f(x)$ such that the tangent lines at the points pass through $(1, 1)$.

Answer:

4. **[15 marks]** Let $f(x) = \dfrac{\sqrt{x}}{e^x}$.

(a) State the domain of $f(x)$. (You do not have to show your work.)

Answer:

(b) Determine if $f(x)$ has any horizontal asymptotes.

Answer:

(c) Determine if $f(x)$ has any vertical asymptotes.

Answer:

(d) Calculate $f'(x)$.

Answer:

(e) Calculate $f''(x)$.

Answer:

(f) Determine the intervals where $f(x)$ is increasing, and the intervals where $f(x)$ is decreasing.

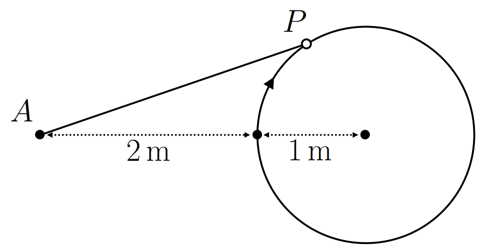
Answer:

(g) Determine the intervals where $f(x)$ is concave up, and the intervals where $f(x)$ is concave down.

Answer:

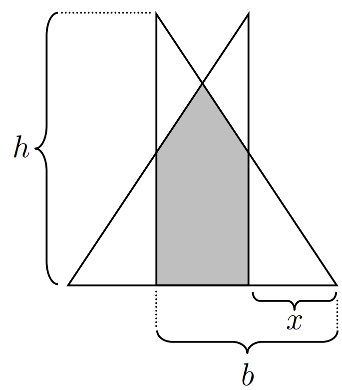
(h) Draw a large sketch of the graph of $f(x)$ below, making sure to include all the features determined in parts (a) through (e).

5. [**10 marks**] Consider a wheel of radius 1 m. A tight elastic band connects the point A, a distance 2 m away from the wheel, to the point P on the wheel. P starts off in the "nearest horizontal position" 2 m away from A. The wheel turns in a clockwise direction at a constant rate of 1 full rotation every 12 seconds. Find the rate at which the length of the elastic band is increasing 2 seconds after the wheel begins to turn.



Answer:

6. [**10 marks**] Consider two right triangles of base b and height h oriented in opposite directions and overlapping, as shown below.



Find x such that the shaded area is maximal. Your answer may be in terms of b or h (or both).

Answer:

7. [**8 marks**] Let $f(x)$ be a differentiable function satisfying

$$f'(x)e^{\cos(x)} - f(x)e^{\cos(x)}\sin(x) = x^{2.1} \quad \text{and} \quad f(0) = 2.$$

(a) Let $g(x)$ and $h(x)$ be differentiable functions. State the formula for the derivative of the product $g(x)h(x)$. (You do not have to show your work.)

Answer:

(b) Find $f(x)$. (Hint: antidifferentiate both sides.)