This examination has 14 pages including this cover.

UBC-SFU-UVic-UNBC Calculus Examination

6 June 2013, 12:00-15:00 PDT

Name: ______________________ **Signature:** ______________________

School: ______________________ **Candidate Number:** ______________________

Rules and Instructions

1. *Show all your work!* Full marks are given only when the answer is correct, and is supported with a written derivation that is orderly, logical, and complete. Part marks are available in every question.
2. Calculators are optional, not required. Correct answers that are "calculator ready," like $3 + \ln 7$ or $e^{\sqrt{2}}$, are fully acceptable.
3. Any calculator compatible with the BC Ministry of Education's *2012/2013 Calculator Policy* may be used.
4. Some basic formulas appear on page 2. No other notes, books, or aids are allowed. In particular, *all calculator memories must be empty when the exam begins.*
5. If you need more space to solve a problem on page n, work on the back of page $n-1$.
6. CAUTION - Candidates guilty of any of the following or similar practices shall be dismissed from the examination immediately and assigned a grade of 0:
   (a) Using any books, papers or memoranda.
   (b) Speaking or communicating with other candidates.
   (c) Exposing written papers to the view of other candidates.
7. Do not write in the grade box shown to the right.

1		42
2		10
3		10
4		10
5		10
6		12
7		6
Total		100

UBC-SFU-UVic-UNBC Calculus Examination

Formula Sheet for 6 June 2013

Exact Values of Trigonometric Functions

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$	π
$\sin\theta$	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0
$\cos\theta$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0	$-1/2$	$-\sqrt{2}/2$	$-\sqrt{3}/2$	-1

Trigonometric Definitions and Identities

$\sin(-\theta) = -\sin\theta$

$\cos(-\theta) = \cos\theta$

$\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \sin\phi\cos\theta$

$\sin 2\theta = 2\sin\theta\cos\theta$

$\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$\sin^2\theta + \cos^2\theta = 1$

$\tan^2\theta + 1 = \sec^2\theta$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\sec\theta = \frac{1}{\cos\theta}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\csc\theta = \frac{1}{\sin\theta}$$

Continued on page 3

[42] **1. Short-Answer Questions.** Write your answers in the boxes provided. Each question is worth 3 marks, but not all questions have equal difficulty. *In this question, and throughout this exam, the answers have no point value unless correct supporting work is also shown.*

(a) Evaluate $\lim_{x \to 9} \dfrac{x\,(\sqrt{x} - 3)}{x - 9}$.

ANSWER:

(b) Find $\int \dfrac{dx}{e^{4x}}$.

ANSWER:

(c) Find y', given $y = (e^{3x} + x)^{2013} \cos(x)$.

ANSWER:

Continued on page 4

(d) Find the derivative of $f(x) = \dfrac{x^7}{\sin(x)}$.

ANSWER:

(e) The function $f(x) = b/(x^2 + ax + 2)$ has a local maximum at $x = 1$, and the local maximum value $f(1)$ equals 2. Find the values of a and b.

ANSWER:

(f) Let $a = \sqrt{100 - (8 \times 10^{-15})}$ and $b = 10 - (4 \times 10^{-16})$. Choose the correct statement and write it in the box: $a < b$, $a = b$, $a > b$. Explain your choice in the work-space provided.

ANSWER:

Continued on page 5

(g) Given that $f'(x) = 2x - (3/x^4)$ for $x > 0$, and $f(1) = 3$, find $f(x)$ for $x > 0$.

ANSWER:

(h) Find the equation of the line that is tangent to the curve $xy^2 + 2xy = 8$ at the point $(1, 2)$.

ANSWER:

(i) Find the (x, y) coordinates of the highest point on the curve $y = \dfrac{8x}{1 + 4x^2}$.

ANSWER:

Continued on page 6

(j) Find all intervals, if any, on which the function below is increasing:

$$f(x) = \frac{1 + \ln(x + 1)}{x + 1}, \quad x > -1.$$

ANSWER:

(k) Find the constant k that satisfies these simultaneous conditions for some function y:

$$\frac{y'}{y} = k, \quad y(0) = 2, \quad y(10) = 100.$$

ANSWER:

(l) A certain function $f(x)$ with $f(1) = 4$ satisfies the identity

$$f'(x) = x\,(f(x))^2 + e^x, \qquad x > 0.$$

Find $f''(1)$.

ANSWER:

Continued on page 7

(m) Given a function $f(x)$ that satisfies $f'(3) = 5$, evaluate this limit or determine that it does not exist:

$$\lim_{x \to 3} \frac{x^2 - 3x}{f(3) - f(x)}.$$

ANSWER:

(n) Find the constant k for which the limit L exists; write the values for both k and L in the answer box:

$$L = \lim_{x \to -\infty} \left(\sqrt{4x^2 + 3x} + kx \right).$$

ANSWER:

Continued on page 8

Full-Solution Questions. In questions 2–7, justify your answers and show all your work. Simplification is not required unless explicitly requested.

[10] **2.** A metal can in the shape of a cylinder with no top must be made to hold 64π cm^3 of liquid. Find the dimensions of the can that minimize the area of the metal required. Be sure to show that your design is a true minimizer.
Hint: The metal used consists of a circle (the bottom of the can) and a rectangle (the sides of the can).

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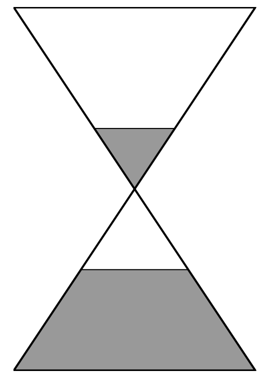
[10] **3.** An idealized hourglass is made by joining two identical circular cones, each with base radius 12 cm and height 18 cm, at their vertices. A tiny hole allows water to drop from the upper cone into the lower one. At a certain instant, the depth of water in the upper cone is 6 cm, this depth is decreasing at a rate of 0.2 cm/min, and the water in the lower cone is 10 cm deep. How fast is the depth of water increasing in the lower cone? (The sketch below shows a snapshot of the situation.)
Hint: The volume of a right circular cone with base radius r and height h is $\frac{1}{3}\pi r^2 h$.



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[10] **4.** The displacement of a point moving along a straight line is given by $s = f(t)$, $t \geq 0$, where t is measured in seconds and s in meters. The point's acceleration function is $a(t) = 10 - 4t$. The point's initial position is $f(0) = 0$, and the point's velocity equals 0 at the instant when $t = 2$.

(a) Find the position function $s = f(t)$.

(b) Find the total distance travelled by the point during the first 3 seconds.

(c) Express the total distance travelled by the point as a function of t, valid for all $t \geq 0$.

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[10] **5.** Consider the plane curve $y = x^3 + 1$. Give this curve the name $\mathcal{C}$.

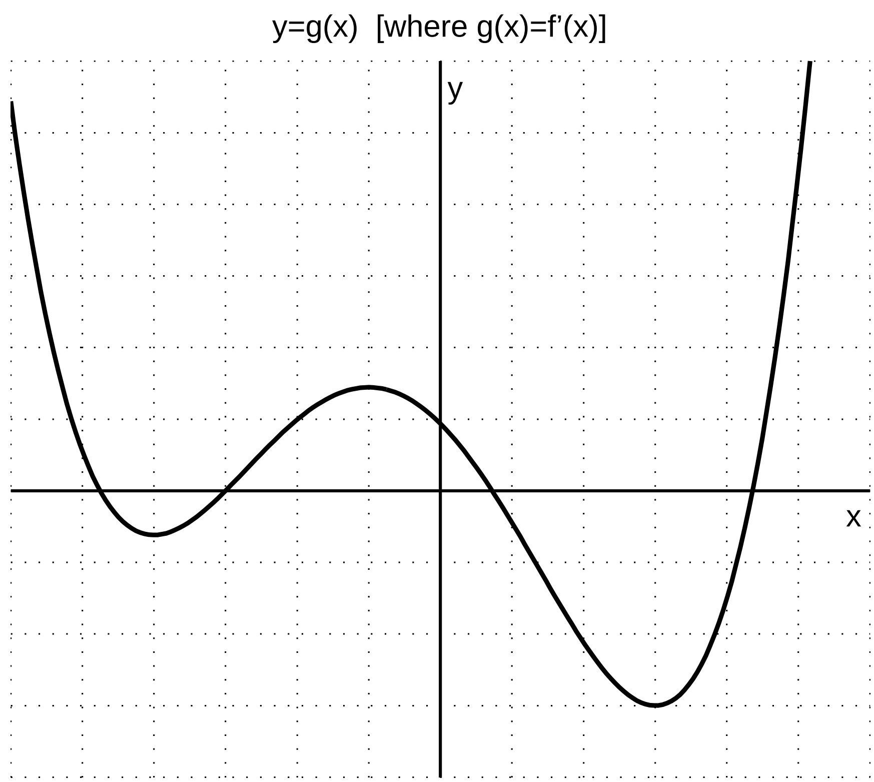
(a) The line that is tangent to $\mathcal{C}$ at the point where $x = 3$ passes through $(a, 0)$. Find a.

(b) We are looking for a point on $\mathcal{C}$ from which the tangent line passes through the point $(2, 0)$. Show how to use Newton's method to find the x-coordinate of such a point, and calculate two Newton steps starting from the initial guess $x = 3$.

(If you have a calculator, report decimal answers. If you don't, present a simplified rational answer for the first step and a "calculator-ready" expression for the second.)

Continued on page 12

[12] **6.** A certain function f satisfies $f(-1) = 0$ and $f'(x) = g(x)$, where the graph of g is shown below. Assume that g is positive-valued and concave up at all points not shown in the sketch. (The spacing between gridlines in the sketch is 1 unit.)



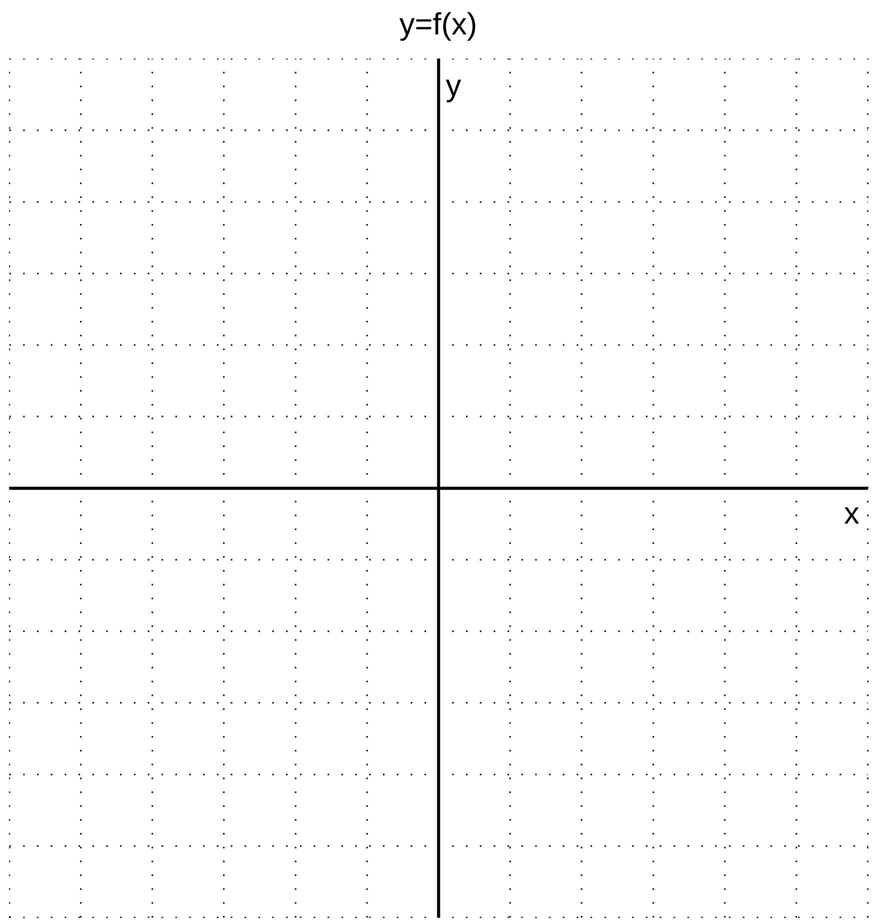
Answer questions (a)–(c) below without attempting to find an algebraic formula for $f(x)$.

(a) Determine the intervals where f is increasing or decreasing, and the (x, y)-coordinates of all local maxima and minima (if any).

Continued on page 13

(b) Determine the intervals where f is concave up or down, and the (x, y)-coordinates of all inflection points (if any).

(c) Sketch the curve $y = f(x)$, showing the features given in items (a)–(b) above and giving the (x, y) coordinates of all points occurring above and also all x-intercepts (if any).



Continued on page 14

[6] **7.** Prove both statements below. Include the names of any famous theorems you rely on.

(a) The equation $x + \ln|x| = 0$ has at least one solution for x in the open interval $(-1, 1)$.

(b) The equation $x + \ln|x| = 0$ has exactly one solution for x in the open interval $(-1, 1)$.

The End