SFU-UBC-UNBC-UVic

Calculus Challenge Exam

June 10, 2010, 12:00-15:00

Name: ______________________________ (please print)

family name *given name*

School: ______________________________

Signature: ______________________________

Question	Maximum	Score
1	6	
2	8	
3	6	
4	6	
5	6	
6	6	
7	10	
8	8	
9	8	
10	6	
11	8	
12	8	
13	8	
14	6	
Total	100	

Instructions:

1. Show all your work. Full marks are given only when the answer is correct, and is supported with a written derivation that is orderly, logical, and complete.
2. A non-graphing, non-programmable calculator which meets ministry standards for the Provincial Examination in Principles of Mathematics 12 may be used. However, calculators are not needed. Correct answers that are calculator ready, like $3 + ln7$ or e^2 , are preferred.
3. A basic formula sheet has been provided. No other notes, books, or aids are allowed. In particular, all calculator memories must be empty when the exam begins.
4. If you need more space to solve a problem, use the back of the facing page.
5. CAUTION - Candidates guilty of any of the following or similar practices shall be dismissed from the examination immediately and assigned a grade of 0:
   (a) Using any books, papers or memoranda.
   (b) Speaking or communicating with other candidates.
   (c) Exposing written papers to the view of other candidates.

1. For each of the following evaluate the limit if it exists or otherwise explain why it does not exist.

[2] (a) $\displaystyle\lim_{x\to 2} \frac{\frac{1}{x} - \frac{1}{2}}{x-2}$

[2] (b) $\displaystyle\lim_{x\to -4^-} \frac{|x+4|}{x+4}$

[2] (c) $\displaystyle\lim_{x\to\infty} \frac{x}{\sqrt{1+2x^2}}$

2. Differentiate each of the following with respect to x.

[2] (a) $y = e^{4x}$

[2] (b) $y = \dfrac{3x - 5}{x^2 + 1}$

[2] (c) $y = x \ln(x^2 + 4)$

[2] (d) $y = \sin(x^2) - \sin^2(x)$

[6] **3. Use the definition of derivative** to find $f'(x)$ where $f(x) = \sqrt{2x + 1}$.

4. Evaluate the following antiderivatives.

[2] (a) $\displaystyle\int \left(\frac{1}{2}x^2 - 2x + 6\right)\,dx$

[2] (b) $\displaystyle\int (3e^x + 7)\,dx$

[2] (c) $\displaystyle\int \left(2\sqrt{x} + 6\cos x\right)\,dx$

[6] **5.** Find the area of the region bounded by the curves $y = x$ and $y = x^2$. **Sketch the graph.**

6. In this question we investigate the solution of the equation

$$2x = \cos x.$$

[3] (a) Explain why you know the equation has **at least** one solution.

[3] (b) Use Newton's Method to approximate the solution of the equation by starting with $x_1 = 0$ and finding x_2.
(Note that you are being asked to find only one iteration of Newton's Method.)

7. Let

$$f(x) = e^{1/x} \qquad f'(x) = -\frac{e^{1/x}}{x^2} \qquad f''(x) = \frac{e^{1/x}(2x+1)}{x^4}$$

[1] (a) What is the domain of f?.

[1] (b) Determine any points of intersection of the graph of f with the x and y axes.

[1] (c) Use limits to determine any horizontal asymptotes of f.

[1] (d) Use limits to determine any vertical asymptotes of f.

[1] (e) For each interval in the table below, indicate whether f is increasing or decreasing.

interval	$(-\infty, 0)$	$(0, \infty)$
$f(x)$		

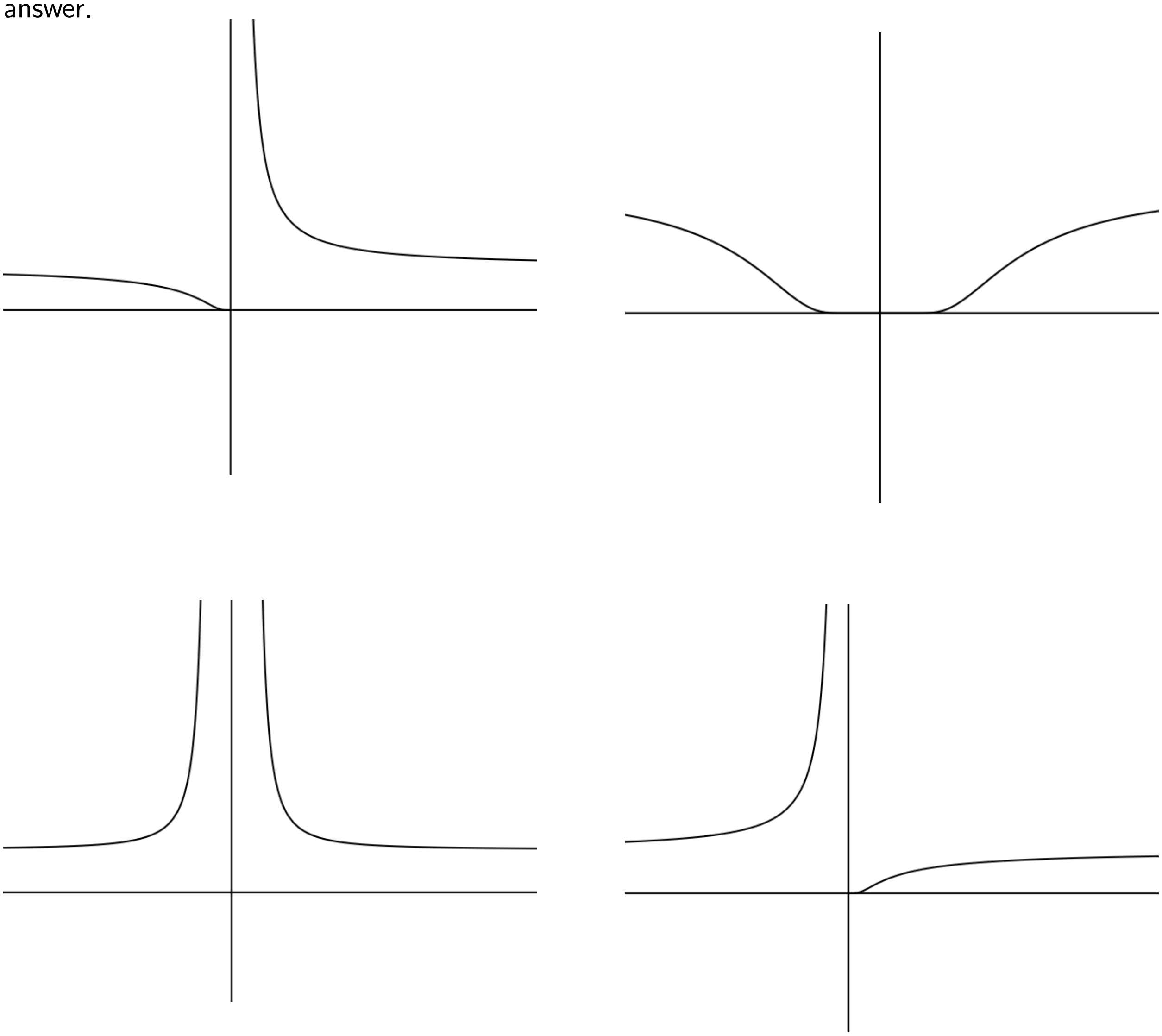
[1] (f) Determine the x coordinates of any local maximum or minimum values of f.

[1] (g) For each interval in the table below, indicate whether f is concave up or concave down.

interval	$(-\infty, -1/2)$	$(-1/2, 0)$	$(0, \infty)$
$f(x)$			

[1] (h) Determine the x coordinates of any inflection points on the graph of f.

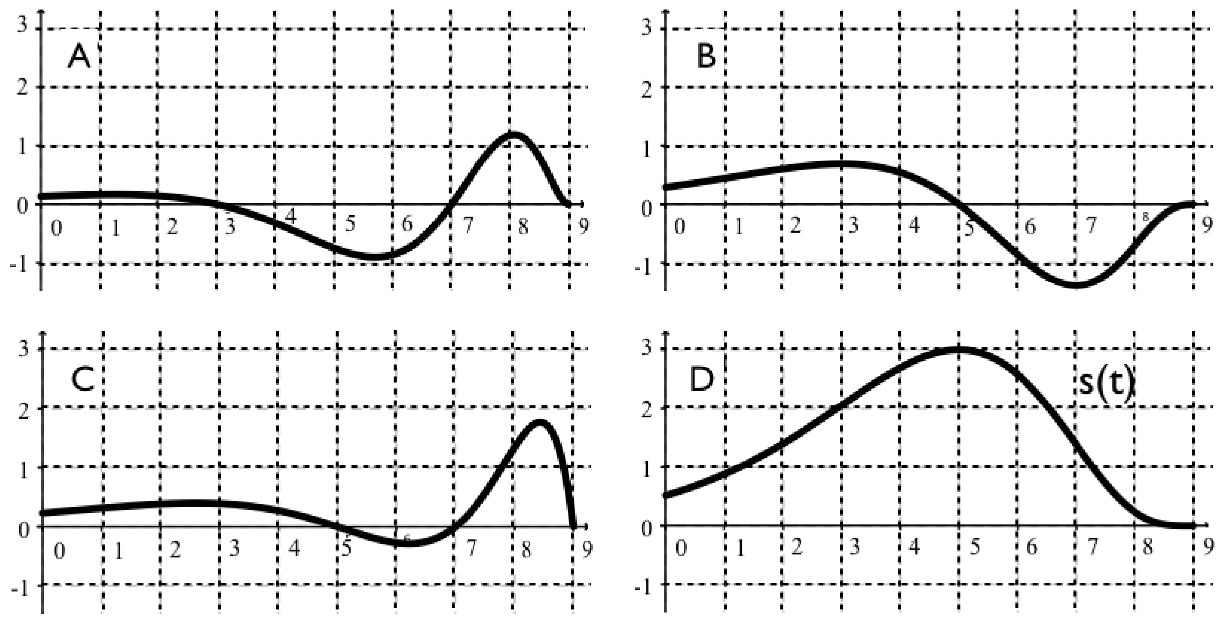
[2] (i) Which of the following best represents the graph of $y = f(x)$? Circle only one answer.



[4] **8.** (a) Suppose that we do not have a formula for $g(x)$ but we know that $g(2) = -4$ and $g'(x) = \sqrt{x^2 + 5}$ for all x. Use a linear approximation to estimate $g(2.05)$.

[4] (b) Is the estimate obtained in part (a) an overestimate or an underestimate of the actual value of $g(2.05)$? [Hint: Consider $g''(x)$.]

9. A particle moves along a line with a position function $s(t)$, where s is measured in meters and t in seconds. Four graphs are shown below: one corresponds to the function $s(t)$, one to the velocity $v(t)$ of the particle, one to its acceleration $a(t)$ and one is unrelated.



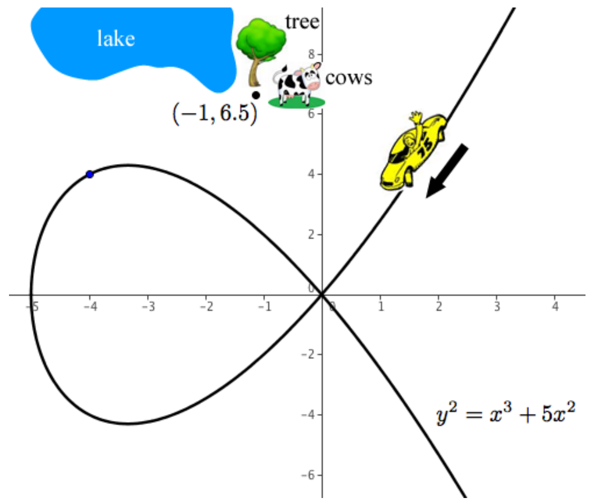
[3] (a) Identify the graphs of $s(t)$, $v(t)$ and $a(t)$ by writing the appropriate letter (A,B,C,D) in the space provided next to the function name. (The position function s is already labeled.)

$s =$ ___*D*___ , $v =$ ____________, $a =$ ____________

[4] (b) Find all time intervals when the particle is slowing down, and when it is speeding up. Justify your answer.

[1] (c) Find the total distance travelled by the particle over the interval $3 \leq t \leq 9$.

10. A race car is speeding around a race-track and comes to a particularly dangerous curve in the shape $y^2 = x^3 + 5x^2$. The diagram below indicates the direction the car is traveling along the curve.



[2] (a) Find the derivative of y with respect to x.

[3] (b) If the car skids off at the point $(-4, 4)$ and continues in a straight path find the equation of the line the car will travel in.

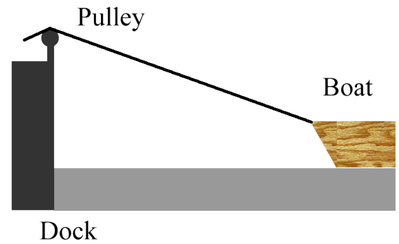
[1] (c) If a tree is located at the point $(-1, 6.5)$ with a lake to the left and cows to the right, will the car hit the lake, the tree or the cows?

11. A cup of coffee, cooling off in a room at temperature $20°$C, has cooling constant $k = 0.09$min^{-1}. Assume the temperature of the coffee obeys Newton's Law of Cooling.

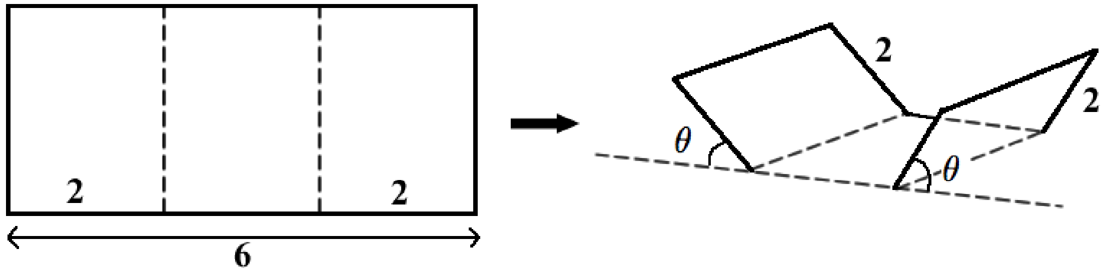
[4] (a) Show that the temperature of the coffee is decreasing at a rate of $5.4°$C/min when its temperature is $T = 80°$C.

[4] (b) The coffee is served at a temperature of $90°$. How long should you wait before drinking it if the optimal temperature is $65°$C? (It is preferred that you leave your answer in the exact form. i.e. as an expression that contains powers of e and/or logarithms.)

[8] **12.** A boat is pulled into a dock by means of a rope attached to a pulley on the dock. The rope is attached to the bow of the boat at a point 1 m below the pulley. If the rope is pulled through the pulley at a rate of 1 m/sec, at what rate will the boat be approaching the dock when there is 10 m of rope between the pulley and the boat?



[8] **13.** A water trough is to be made from a long strip of tin 6 ft wide by bending up at an angle θ a 2 ft strip at each side. What angle θ would maximize the cross sectional area, and thus the volume, of the trough?



[6] **14.** Find a function f such that $f'(x) = x^3$ and the line $x + y = 0$ is tangent to the graph of f.

Formula Sheet

Exact Values of Trigonometric Functions

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$	π
$\sin\theta$	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0
$\cos\theta$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0	$-1/2$	$-\sqrt{2}/2$	$-\sqrt{3}/2$	-1

Trigonometric Definitions and Identities

$$\sin(-\theta) = -\sin\theta$$

$$\cos(-\theta) = \cos\theta$$

$$\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \sin\phi\cos\theta$$

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

$$\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$

$$\sin^2\theta = \frac{1-\cos(2\theta)}{2}$$

$$\cos^2\theta = \frac{1+\cos(2\theta)}{2}$$

$$\sin^2\theta + \cos^2\theta = 1$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\sec\theta = \frac{1}{\cos\theta}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\csc\theta = \frac{1}{\sin\theta}$$