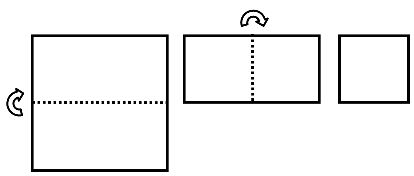
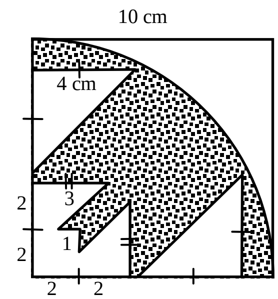
UBC Grade 8–10 Workshop Problems 2006

1. If I drive 80 km/h for 2.5 hrs, then change speed and drive for another 1.5 hours, and if I traveled 380 km in total, how fast was I going for the second part of my trip?

2. Superman misinterprets the cubism movement and paints a large solid cubic sculpture blue. Just as he finishes, Spiderman jealously breaks his blue cube sculpture into 1000 perfect little equal-sized cubes and puts them all into a large bag. If Superman reaches into the bag and pulls out one of those little cubes, what is the probability that he will pull out one painted blue on exactly two sides?

3. A piece of square paper of side length 20 cm is folded in half twice, and a snowflake is cut out as shown so that its outer boundary is a circle. What is the total area of the snowflake?





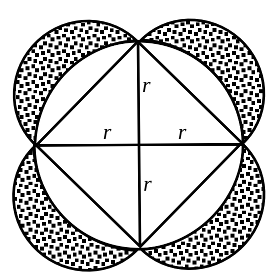
4. To find the height of a building Larry stands 8 m away from a 7.5-m high pole that is 48 m from the building. Larry's eyes are 1.5 m from the ground. At this position he discovers that he can align the top of the pole with the top of the building. How high is the building?

5. Danielle has a very unusual 12-hour watch. Every time the hour hand passes three even numbers, it immediately jumps back to the previous prime number; then the watch continues until it has again passed three even numbers at which point it jumps back to the previous prime number, etc. If Danielle's watch is started at 11 a.m. today, and reads the correct time then, what is the actual time when her watch next reads 11 o'clock? After that time, when will the watch first read the correct time?

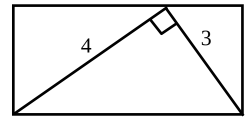
6. Wonderwoman is 1400 km away from Metrotown, and Superman is 600 km away from Metrotown, in the opposite direction. The two superheroes fly to Metrotown at speeds of 200 km/h and 100 km/h respectively. Superfly starts on Wonderwoman and flies back and forth between the superheroes at 300 km/h, stopping when it lands on one of our heroes at Metrotown. What is the total distance Superfly travels before stopping at Metrotown?

7. Daniel and Willy race around an oval track with a circumference of 400 m. They start at the same place on the track but run in opposite directions, each at a constant rate. The winner is the first one who runs 4 km. If Willy runs at 2/3 the speed of Daniel, how many times will the two runners cross paths during the race?

8. In the diagram, each of the four semicircles has the corresponding side of the square as its diameter. Find the area of the shaded region if the large circle has radius $r = 4$.



9. In the rectangle, we can draw a right triangle of side lengths 4 and 3 as shown. What are the dimensions of the rectangle?



10. Nemo, a juvenile clownfish, is trapped in the aquarium of a Dr. P. Sherman, a dentist. The aquarium tank is 100 cm long, 50 cm high, and 20 cm wide, and the water level is now at the 88% level. Water evaporates from the tank at a rate of 0.5 cm^3/day. Each day, Dr. Sherman puts the teeth he extracts into the tank. Each tooth has a volume of 0.1 cm^3. Nemo can escape from the tank when the water level is within 5 cm of the top. If it is now Monday morning, and Dr. Sherman pulls 40 teeth on Mondays, 30 on Tuesdays, none on Wednesdays, 45 on Thursdays, 20 on Fridays, and none on weekends, on what day of the week will Nemo escape?

11. Mr. Brown wins the $60-million jackpot in Lotto 649. He has 10 friends with whom he shares the prize. He gives 1/11 of the jackpot to his best friend, 1/12 of the remaining money to his second-best friend, 1/13 of what is left to his third-best friend, etc. How much money does Mr. Brown keep for himself?

12. Mrs. Williams wins the Lotto 21 jackpot. The prize is paid entirely in loonies and toonies. She gives all the money away to her grandchildren, ensuring that each child receives the same number of coins. Victoria receives 1/3 of the loonies and 1/12 of the toonies; Jordan receives 1/8 of the loonies and 1/2 of the toonies. How many grandchildren does Mrs. Williams have?

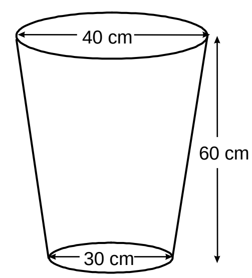
13. Four friends decide to play gin rummy for money in the following way. Each player starts with $6. After each game of rummy, everyone except the winner puts $1 into the pot. Each player continues playing until he or she has run out of money. The pot keeps growing until only one player is remaining, and then that player wins the entire pot. What is the maximum number of games needed to determine the winner?

14. Consider the sequence created by writing the numbers from 1 to 9999 in order (the sequence starts 123456789101112…).

a) Find the 2006th digit in the sequence.

b) Find the total number of zeroes in the sequence.

15. Consider a trash can that is shaped like a truncated cone. Its top has a diameter of 40 cm, its bottom has a diameter of 30 cm, and its height is 60 cm. Find the radius of the largest ball that will touch the bottom of the trash can if tossed into it.



16. Five people decide to share a storage locker. They put a number of combination locks on the locker in such a way that the locker can only be opened if each of the locks is unlocked. The five people don't entirely trust one another, so they agree to secure it with enough locks so that the locker can be opened when, and only when, a majority of them are present. What is the fewest number of locks they will need?

SOLUTIONS

Note: These concise solutions are meant for workshop leaders and teachers. Presentations to Grade 8–10 students should include additional detail and motivation. The solutions outlined here are considered appropriate for school students at this level; alternate solutions are often possible.

1. Answer: 120 km/h. During the first 2.5 hours, I drove $2.5 \times 80 = 200$ km. So, I covered $380 - 200 = 180$ km in the final 1.5 hours. I was going $180/1.5 = 120$ km/h for the second part of the trip.

2. Answer: 96/1000. Of the 1000 little cubes, the ones painted blue on exactly two sides are those along an edge but not at a corner. There are 12 edges and 8 such cubes along each edge, so $12 \times 8 = 96$ two-blue-sided little cubes. Assuming randomness of Superman's extraction, the probability is 96/1000 (a.k.a. 12/125).

3. Answer: $100\pi - 112$. Use symmetry. The total area of the snowflake is 4 times the area shaded in the diagram. The quarter circle has area $(1/4)\cdot\pi(10)^2$ cm^2. From this area we subtract the area of the interior white regions. The two large triangles form a square of area 4^2 cm^2 if they are pushed together. The 8-sided white region is itself a square of area 4^2 cm^2 less a 6-sided region that can be rearranged into a rectangle of side lengths $2^{1/2}$ cm and $2\cdot 2^{1/2}$ cm, by, for example, Pythagoras. So, multiplying by 4, the total area is $100\pi - 4(16 + 16 - (2^{1/2})(2\cdot 2^{1/2})) = 100\pi - 112$. There are many ways to compute the area; this one only requires knowledge of the Pythagorean Theorem (or the 45-45-90 right triangle) and the area of a rectangle.

4. Answer: 43.5 m. Draw a diagram, and then draw a horizontal line from Larry's eyes to the building. We see two similar triangles, one of width 8 m and height $7.5 - 1.5 = 6$ m and the other of width $8 + 48 = 56$ m and height 1.5 m less than the height of the building. This diminished height is $56 \times (6/8) = 42$ m, so the building has height $42 + 1.5 = 43.5$ m. Alternatively, extend the line from the top of the building to the top of the pole (and Larry's eyes) to the ground, using similar triangles deduce that this line hits the ground 2 m away from Larry's feet, and then do the calculation $1.5 \times (58/2) = 43.5$. Some students may attempt to use "similar quadrilaterals," but that approach will yield an incorrect answer.

5. Answers: 1 a.m.; 11 p.m. two days from now (60 hours from 11 a.m. today). When the watch reaches 4 p.m., it will jump back 1 hour, so that it reads 3 o'clock even though it is actually 4 p.m. Now it continues till it reads 8 o'clock, when it is actually 9 p.m., and jumps back 1 more hour, so that it is reading 7 o'clock at 9 p.m. In 4 more hours, i.e. at 1 a.m., it will read 11 o'clock. For the second part of the problem, continue running the clock till it reads 12 o'clock, at 2 a.m. The clock then jumps back to 11 o'clock again. So, it has lost 3 hours, and we are back at our initial state in which the clock reads 11 o'clock and moves forward to reading 4 o'clock, 3 o'clock, etc. After 1 more 15-hour cycle, the clock will have lost another 3 hours and be back at its initial state again; after 2 additional cycles (60 actual hours altogether), the clock will correctly read 11 o'clock. Since the final hour lost in each 15-hour cycle is lost at the very end of the cycle, this is the earliest time at which the clock first reads the correct time again.

6. Answer: 2000 km. The relative speeds of Superfly with respect to Superman and Wonderwoman are $300 + 100 = 400$ km/h and $300 + 200 = 500$ km/h respectively: The initial separation between the superheroes is $1400 + 600 = 2000$ km, and the first relative speed means Superfly meets Superman after $2000/400 = 5$ hours, having flown $5 \times 300 = 1500$ km. At this time, Wonderwoman is $1400 - 5 \times 200 = 400$ km away from Metrotown and Superman is $600 - 5 \times 100 = 100$ km away, and the superheroes are $400 + 100 = 500$ km apart. Since the second relative speed is 500 km/h, Superfly, after reversing direction, lands on Wonderwoman 1 hour later; at this time Superman has arrived at Metrotown and Wonderwoman is $400 - 200 = 200$ away. Superfly has now flown $1500 + 300 = 1800$ km, and after flying the 200 km to Metrotown has completed its journey. The problem may be solved instead by considering what happens during various discrete time intervals; the numbers here are chosen so that only multiples of an hour need to be invoked. Draw pictures!

7. Answer: 16. Consider each of Daniel's 10 laps before he wins the race. On the first lap, he crosses Willy once, and at the end of this lap Willy is 1/3 of a lap away from the starting point, where Daniel is, heading toward Daniel. On Daniel's second lap, he crosses Willy early in the lap, and then again later on; on Daniel's third lap he crosses Willy twice, including the crossing at the very end of the lap. So, during Daniel's first 3 laps there are a total of 5 crossings. We are back where we started (periodicity again – see question 5!) and repeat. After 9 laps there have been 15 crossings, and then there is one final crossing during Daniel's 10th lap. Alternate solution: Daniel meets Willy every 3/5 of a lap he (Daniel) runs (explain why), and $10/(3/5)$ has integral part 16. A diagram helps here.

8. Answer: 32. Keeping "r" in the problem, the square has side length $2^{1/2}\cdot r$. The area of the region outside the square and inside the large circle is thus $\pi r^2 - (2^{1/2}\cdot r)^2 = (\pi - 2)r^2$. The shaded region has area twice the area of a circle of radius $(2^{1/2}\cdot r)/2$ less this quantity, i.e. $2\cdot\pi\cdot((2^{1/2}\cdot r)/2)^2 - (\pi - 2)r^2 = 2r^2$, which gives 32 if $r = 4$. Note the final answer involves no πs. Students who don't retain *exact* intermediate expressions will obtain approximations to 32.

9. Answer: 5 by 12/5. By Pythagoras or common knowledge, the width of the rectangle is 5. If h is the height of the rectangle, then computing the area of the right triangle in two ways gives $(4)(3)/2 = (5)(h)/2$, so $h = 12/5$. There are other, more algebraic, ways to compute h, but this method is quickest.

10. Answer: Friday. Consider one entire week. A total of 135 teeth are extracted, increasing the volume of the tank's contents by 13.5 cm^3. Evaporation reduces the volume by 3.5 cm^3, so over the course of an entire week the volume increases by 10 cm^3. Now since the water level starts at 44 cm (88% of 50) and need to rise to 45 cm, the volume must increase by $100 \times 20 = 2000$ cm^3. This exact increase will have occurred the Monday morning 200 weeks from now. The preceding Friday morning the volume will be $2 – 1.5 = 0.5$ cm^3 lower. Since a day's evaporation is also 0.5 cm^3, escape cannot occur the day before. Friday's extractions increase the volume by 2 cm^3, so Nemo will escape on this Friday. That assumes that Nemo's own volume has not increased in 4 years!

11. Answer: \$30 million. After giving his best friend 1/11 of the jackpot, Mr. Brown has 10/11 remaining. After giving the second-best friend 1/12, he as 11/12 of 10/11 remaining, i.e. $(11/12) \times (10/11) = 10/12$ remaining. Continue, multiplying by 12/13, 13/14,…, 19/20. Noting the cancellation, we see that Mr. Brown retains 10/20 or 1/2 of the jackpot.

12. Answer: 4. First note that the number of loonies must be divisible by 24, since 24 is the least common multiple of 3 and 8, and similarly the number of toonies must be divisible by 12. Moreover, since $(1/3) \times 24 + (1/12) \times 12$ equals 9 and so does $(1/8) \times 24 + (1/2) \times 12$, a jackpot of 24 loonies and 12 toonies fits the bill, and that gives $(24 + 12)/9 = 4$ grandchildren. In fact the jackpot can be any multiple of this configuration (e.g. 48,00,000 loonies and 24,000,000 toonies to make our meagre Lotto 21 more lucrative than problem 11's Lotto 649), but no matter the multiple there will still be 4 grandchildren.

13. Answer: 11. It'll take at least 6 games before any player runs out of money, so for each of the first 6 games the pot grows by \$3. At the end of the 6th game, there is just $4 \times \$6 – \$18 = \$6$ still in the hands of the 4 players. The maximum number of games will result from having the pot grow most slowly thereafter, and with at least two players staying alive as long as possible. If the \$6 is distributed among two players each having \$3 (so that the other two players lost all the games and each of these two players won 3 games), then the pot grows by just \$1. And if the two players take turns winning, so they both last as long as possible, then 5 additional games are needed.

14. Answers: 0 and 2889. For a), consider blocks of 1-digit, 2-digit, 3-digit, and 4-digit numbers. The first block takes us to the 9th digit of the sequence, the second to the $9 + 2\cdot 90 = 189$th, the third to the $189 + 3\cdot 900 = 2889$th. So, the 2006th digit occurs somewhere in the third block. To be precise, $2006 – 189 = 1817$, which gives a quotient of 605 and remainder 2 when divided by 3, so the 2006th digit in the sequence is the 2nd digit of 705, or 0. For b), 999 numbers have a 0 in the 1s place (numbers of the form $10n$ where n ranges from 1 to 999), 990 have a 0 in the 10s place ($100n + m$ where n ranges from 1 to 99 and m ranges from 0 to 9), 900 have a 0 in the 100s place ($1000n + m$ where n ranges from 1 to 9 and m from 0 to 99), and none have a 0 in the 1000s place. So, there are a total of $999 + 990 + 900 = 2889$ 0s.

15. Answer: $(5/4)\cdot(145^{1/2} + 1)$ cm. Let the radius of the largest ball be r. Taking a vertical cross-section through the middle of the trash can and placed ball, we have a circle of radius r that is tangent to the bottom horizontal line of length 30 and the sloped side line, which we call L. The radius from the centre of the circle to L is perpendicular to L. Now, extend L so that it meets the vertical line through the centre of the trash can and ball; using the given dimensions and easy similarity the intersection occurs 180 cm below the bottom of the can. There are two similar right triangles, the smaller of which has hypotenuse $180+r$ and one side r and the larger of which has hypotenuse $(20^2 + 240^2)^{1/2}$ and corresponding side 20. The similarity equation simplifies to $(180+r)/r = 145^{1/2}$, which yields $r = 180/(145^{1/2} – 1)$. This rationalizes to the given answer.

16. Answer: 10. Suppose only 2 people — call them A and B — are present. Then at least one of the combinations is not known to either, and hence is known to at most the other 3 — call these people C, D, and E. But if 1 of these 3 people does not know this combination, say C, then A, B, and C would constitute a majority that could not open this lock and hence could not open the locker. So in fact C, D, and E all know this combination. The choice of A and B was arbitrary, so we've shown that for each choice of 3 of the 5 people, some combination is known by exactly these 3. There are 10 such choices, hence at least 10 locks. The reasoning above quickly shows that in fact these 10 locks are sufficient.