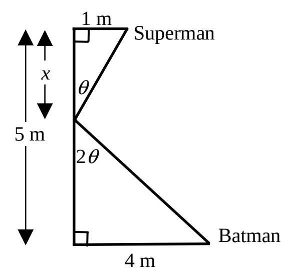
UBC Grade 11–12 Workshop Problems 2006

1. Daniel and Willy race around an oval track with a circumference of 400 m. They start at the same place on the track but run in opposite directions, each at a constant rate. The winner is the first one who runs 4 km. If Willy runs at 2/3 the speed of Daniel, how many times will the two runners cross paths during the race?

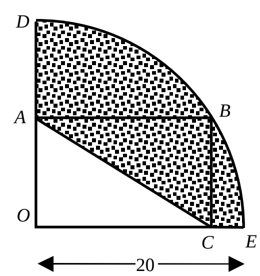
2. Find all integer solutions (n, m) to the equation $(n-2)^{m^2-4} = 1$.

3. An equilateral triangle is inscribed in a circle, which is itself inscribed in a square, which is in turn inscribed in another circle of radius 4. What is the length of the side of the triangle?

4. Superman and Batman are playing a game of Superball. Superball involves a freestanding superwall of length 5 m. Our heroes stand at opposite ends of the wall and are 1 m and 4 m perpendicularly from the wall respectively, as shown. Because of the superball's superspin, the angle at which it rebounds from the wall after being thrown by Superman is twice the incoming angle. At what distance x from his end of the wall must Superman throw the ball so that it banks off the wall and hits Batman? Superman is also super at math: he knows that $\tan 2\theta = (2\tan\theta)/(1-\tan^2\theta)$.



5. The quarter circle shown at the right has centre O and radius 20. If the area of triangle OAC is 69, what is the perimeter of the shaded region?



6. Consider the sequence created by writing the numbers from 1 to 9999 in order (the sequence starts 123456789101112…). Find the total number of zeroes in the sequence.

7. A bag of marbles contains 8 black marbles and 9 white marbles. Marbles are randomly drawn from the bag one by one.

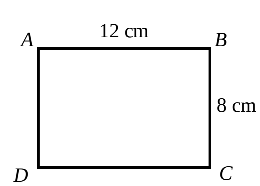
a) Suppose that after a marble is drawn it is placed back into the bag. What is the probability that the second marble picked is black?

b) Suppose that after a marble is drawn it is *not* placed back into the bag. What is the probability that the second marble picked is black?

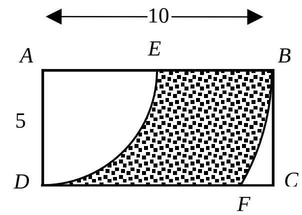
What do you notice about your answers to parts a) and b)?

8. Four friends play gin rummy for money in the following way. Each player starts with \$6. After each game of rummy, everyone except the winner puts \$1 into the pot; for example, after the first game the pot has \$3 in it. Each player continues playing until he or she has run out of money. The pot keeps growing until only one player is remaining, and then that player wins the entire pot. What is the maximum number of games needed to determine the winner?

9. *ABCD* is the rectangle shown at the right. It is folded so that the corner *D* meets the midpoint of the side *AB*. Find the length of the fold.



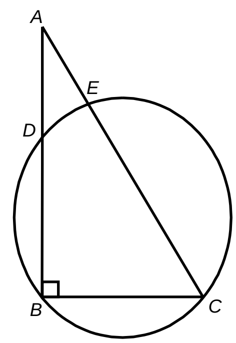
10. Find the area of the shaded region. *ABCD* is a rectangle with *AD* = 5 and *AB* = 10, and *A* is the centre of the indicated circular arcs.



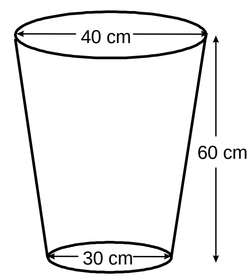
11. Find the number of distinct pairs of positive integers whose product is 2006 times their sum. (The order of the numbers is irrelevant; for example, we would consider (2242, 19057) and (19057, 2242) to be the same pair of numbers.) Note that the factorization of 2006 into prime numbers is $2006 = 2 \times 17 \times 59$.

12. If $x = (3+\sqrt{7})/2$ and $y = (3-\sqrt{7})/2$, find $16x^4 + 30x^2y^2 + 16y^4$.

13. In the diagram, the circle has radius 6, angle *DBC* equals 90°, points *B*, *C*, *D*, and *E* lie on the circle, *AD* has length 5, and *AE* has length 4. Find the length of *BC*.



14. Consider a trash can that is shaped like a truncated cone. Its top has a diameter of 40 cm, its bottom has a diameter of 30 cm, and its height is 60 cm. Find the radius of the largest ball that will touch the bottom of the trash can if tossed into it.



15. Five people decide to share a storage locker. They put a number of combination locks on the locker in such a way that the locker can only be opened if each of the locks is unlocked. The five people don't entirely trust one another, so they agree to secure it with enough locks so that the locker can be opened when, and only when, a majority of them are present. What is the fewest number of locks they will need?

16. Consider a rectangle of height 3 and width 10. The rectangle has area 30, equal to the total area of fifteen 1×2 rectangles. A *tiling* is an arrangement of the small indistinguishable rectangles (called *tiles*) that completely covers the large rectangle without any overlap.

a) Explain why there is no tiling in which exactly 1 tile is vertical (height 2 and width 1) and the other 14 are horizontal.

b) How many tilings are there in which exactly 2 tiles are vertical and the other 13 are horizontal?

c) What is the total number of tilings?

SOLUTIONS

Note: These concise solutions are meant for workshop leaders and teachers. Presentations to Grade 11/12 students should include additional detail and motivation. The solutions outlined here are considered appropriate for school students at this level; alternate solutions are often possible.

1. Answer: 16. Consider each of Daniel's 10 laps before he wins the race. On the first lap, he crosses Willy once, and at the end of this lap Willy is 1/3 of a lap away from the starting point, where Daniel is, heading toward Daniel. On Daniel's second lap, he crosses Willy early in the lap, and then again later on; on Daniel's third lap he crosses Willy twice, including the crossing at the very end of the lap. So, during Daniel's first 3 laps there are a total of 5 crossings. We are back where we started and repeat. After 9 laps there have been 15 crossings, and then there is one final crossing during Daniel's 10th lap. Alternate solution: Daniel meets Willy every 3/5 of a lap he (Daniel) runs (explain why), and 10/(3/5) has integral part 16. A diagram helps here.

2. Answer: (3, m) (m arbitrary); (n, 2) (n arbitrary); (n, –2) (n arbitrary); and (1, m) (m even). If a and b are integers, then $a^b = 1$ precisely when $a = 1$ with b arbitrary, $b = 0$ with a arbitrary, or $a = -1$ and b even. Note that we are adopting the convention $0^0 = 1$ so that the case $a = 0$, $b = 0$ is included in our solution set; if we say instead that 0^0 is "undefined" then we would drop this case. Noting that $m^2 - 4$ is even if and only if m is even, we conclude that the solutions are: (3, m) (m arbitrary); (n, 2) (n arbitrary); (n, –2) (n arbitrary); and (1, m) (m even) (with the solutions (2, 2) and (2, –2) dropped if we regard 0^0 as undefined). Students may miss the final class of solutions.

3. Answer: $2\sqrt{6}$. A 45-45-90 right triangle with hypotenuse running between the common centre of the various figures and a vertex of the square shows that half the square's has side length, which equals the smaller circle's radius, is $2\sqrt{2}$. A similarly placed 30-60-90 right triangle shows that half the triangle's side length is $(2\sqrt{2})(\sqrt{3}/2) = \sqrt{6}$. An exercise in trigonometry, with emphasis on the standard right triangles.

4. Answer: $x = 2$ m. Using the two indicated right triangles and Superman's given trigonometric knowledge, we arrive at the equation $4/(5 - x) = (2(1/x)/(1 - (1/x)^2))$, which upon algebraic manipulation becomes $3x^2 - 5x - 2 = 0$. The solutions are –1/3 and 2, the former of which is clearly not allowed. Note that it's easy enough to instead guess the correct answer from the diagram and verify that the guess is correct.

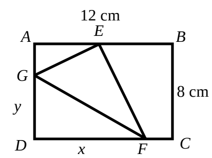
5. Answer: $10\pi + 34$. A crucial observation is that AC has length 20 since the two diagonals of a rectangle have the same length and the other diagonal is a radius of the circle. Now let a and c respectively be the lengths of OA and OC. We are given the equation $(ac/2) = 69$, and using Pythagoras we have $a^2 + c^2 = 400$. We could easily enough solve for a and c, but instead note, by starting at A and going counterclockwise, that the required perimeter equals $20 + (20 - c) + 2\pi(20)/4 + (20 - a) = 10\pi + 60 - (a + c)$. Using the identity $(a + c)^2 = a^2 + c^2 + 2ac$ we calculate: $(a + c)^2 = 400 + 4\cdot 69 = 676 = 26^2$, yielding the given answer.

6. Answer: 2889. 999 numbers have a 0 in the 1s place (numbers of the form $10n$ where n ranges from 1 to 999), 990 have a 0 in the 10s place ($100n + m$ where n ranges from 1 to 99 and m ranges from 0 to 9), 900 have a 0 in the 100s place ($1000n + m$ where n ranges from 1 to 9 and m from 0 to 99), and none have a 0 in the 1000s place. So, there are a total of 999 + 990 + 900 = 2889 0s.

7. Answers: 8/17 and 8/17. Part a) is routine (and meant only to lead students into part b)). For part b), consider two cases: 1) the first marble picked is black; 2) the first marble picked is white. Since in case 1) 7 of the 16 marbles remaining for the second selection are black, the probability of this BB branch is (8/17)(7/16). For case 2), we require a WB branch, which has probability (9/17)(8/16). When these two products are added together, the sum magically reduces to 8/17, which is the *same* answer as in part a). Students who are intrigued by this problem might be asked to what extent this equality generalizes. For example, does it depend on the number of marbles of each colour, and is it also true for the third marble picked? Is there some underlying principle at play here?

8. Answer: 11. It'll take at least 6 games before any player runs out of money, so for each of the first 6 games the pot grows by \$3. At the end of the 6th game, there is just 4 × \$6 – \$18 = \$6 still in the hands of the 4 players. The maximum number of games will result from having the pot grow most slowly thereafter, and with at least two players staying alive as long as possible. If the \$6 is distributed among two players each having \$3 (so that the other two players lost all the games and each of these two players won 3 games), then the pot grows by just \$1. And if the two players take turns winning, so they both last as long as possible, then 5 additional games are needed.

9. Answer: 125/12 cm. Complete the diagram as shown, so that the fold line is FG and E is the midpoint of AB. Let x be the length in cm of DF and y the length of DG. Using the equality of the lengths of DG and GE and Pythagoras, we have $y^2 = (8-y)^2 + 6^2$, which gives $y = 25/4$. The equality of the lengths of DF and FE gives $x^2 = (x - 6)^2 + 8^2$, hence $x = 25/3$. Finally, by Pythagoras GF has length $((25/4)^2 + (25/3)^2)^{1/2} = 25((3^2 + 4^2)/(4\cdot 3)^2)^{1/2} = 25\cdot(5/12) = 125/12$. There are many ways to parameterize this problem.



10. Answer: $(25/2)\cdot 3^{1/2} + (25/12)\pi$. Draw the line from *A* to *F*. This line has length 10 and the angle *BAF* is 30 degrees since its sine is 5/10 = 1/2. The circular sector *AFB* therefore has area $(30/360)(\pi\cdot 10^2) = (25/3)\pi$. Since by Pythagoras *DF* has length $75^{1/2} = 5\cdot 3^{1/2}$, triangle *ADF* has area $(25/2)\cdot 3^{1/2}$. Adding these two areas and subtracting the area of the quarter circular sector *AED*, which is $\pi\cdot 5^2/4 = (25/4)\pi$, we obtain the given answer.

11. Answer: 14. We need to find the number of positive-integer solutions to the equation $ab = 2006(a + b)$. We can rewrite this equation as $(a - 2006)(b - 2006) = 2006^2 = 2^2 \times 17^2 \times 59^2$. The number on the right hand side of this equation has 3 × 3 × 3 = 27 factors. Of these, 26 occur in 13 pairs, and 1, 2006, in a singleton. This yields 13 + 1 = 14 solutions, after we note that *a* and *b* must each be larger than 2006.

12. Answer: 2047/2. Write the given quartic expression as $(4x^2 + 4y^2)^2 - 2x^2y^2 = ((2x + 2y)^2 - 8xy)^2 - 2(xy)^2$ and use the facts that $xy = 1/2$ and $x + y = 3$.

13. Answer: $3\cdot (4 + 135^{1/2})/5$. Since angle *DBC* is a right angle, the line *DC* is a diameter of the circle, and hence the angle *DEC* is also a right angle. So the angle *DEA* is as well, and Pythagoras tells us that *DE* has length 3. From Pythagoras again, *EC* has length $(12^2 - 3^2)^{1/2} = 135^{1/2}$. Finally, using the similarity of triangles *AED* and *ABC* we conclude that the length of *BC* is $3\cdot (4 + 135^{1/2})/5$.

14. Answer: $(5/4)\cdot(145^{1/2} + 1)$ cm. Let the radius of the largest ball be *r*. Taking a vertical cross-section through the middle of the trash can and placed ball, we have a circle of radius *r* that is tangent to the bottom horizontal line of length 30 and the sloped side line, which we call *L*. The radius from the centre of the circle to *L* is perpendicular to *L*. Now, extend *L* so that it meets the vertical line through the centre of the trash can and ball; using the given dimensions and easy similarity the intersection occurs 180 cm below the bottom of the can. There are two similar right triangles, the smaller of which has hypotenuse $180+r$ and one side *r* and the larger of which has hypotenuse $(20^2 + 240^2)^{1/2}$ and corresponding side 20. The similarity equation simplifies to $(180+r)/r = 145^{1/2}$, which yields $r = 180/(145^{1/2} - 1)$. This rationalizes to the given answer.

15. Answer: 10. Suppose only 2 people — call them A and B — are present. Then at least one of the combinations is not known to either, and hence is known to at most the other 3 — call these people C, D, and E. But if 1 of these 3 people does not know this combination, say C, then A, B, and C would constitute a majority that could not open this lock and hence could not open the locker. So in fact C, D, and E all know this combination. The choice of A and B was arbitrary, so we've shown that for each choice of 3 of the 5 people, some combination is known by exactly these 3. There are 10 such choices, hence at least 10 locks. The reasoning above quickly shows that in fact these 10 locks are sufficient.

16. Answers: b) 30; c) 571. Place a unit grid onto the large rectangle ("rectangle" for short). The tiles must all jibe with this grid. For a), note that on one side of a sole vertical tile there would be an odd number of unit squares in some row of the rectangle, and it would be impossible to cover this odd number of squares with horizontal tiles. For b), note that the 2 vertical tiles must be horizontally offset, or else we would again have some row with an odd length. Moreover, there must be an even distance between the left edge of the left vertical tile and the left edge of the rectangle, between the right edge of the left vertical tile and the left edge of the right vertical tile, and between the right edge of the right vertical tile and the right edge of the rectangle. This restricts the vertical tiles to the following 15 pairs of columns: (1,2), (1,4), (1,6), (1,8), (1,10), (3,4), (3,6), (3,8), (3,10), (5,6), (5,8), (5,10), (7,8), (7,10), (9, 10). Since the vertical tiles can either be in the first two rows or the last two, we double this number for our final answer. Using the notation *n*_C_*r* to denote "*n* choose *r*," the answer to c) equals $(5\_C\_0)\cdot 2^0 + (6\_C\_2)\cdot 2^1 + (7\_C\_4)\cdot 2^2 + (8\_C\_6)\cdot 2^3 + (9\_C\_8)\cdot 2^4 + (10\_C\_10)\cdot 2^5$. Note for example that the second term in this sum equals 30, the answer to part b). Bright students should be challenged into discovering the solution, perhaps by being simply told the answer in this structural form and then being asked to explain it. It's a matter of understanding how to properly generalize the answer to part b).