UBC Workshops 2005
Solutions B

1. At a banquet, there was one serving dish of rice for every three people, one serving dish of vegetables for every four people, and 2 serving dishes, one of duck and one of fish, for every six people. Altogether, 88 serving dishes of food were set out. How many people were at the banquet?

Solution. Banquet problems of this type have been part of the Chinese problem literature since -200, and may be much older than that.

We could "guess" that there were 12 people at the banquet. Of course the guess is too small. But it is convenient, since 12 is divisible by 3, 4, and 6.

Let's check whether the guess is right. If there really were12 people at the banquet, they had 4 serving dishes of rice, 3 of vegetables, and 2 each of duck and fish, for a total of 11. Wrong guess, there were actually 88 dishes of food served.

But note that $88/11 = 8$, so if we multiply the "guess" by 8, we get the right answer. We conclude that there were 96 people at the banquet. So by thinking about one guess that turned out to be wrong, we can go directly to the right answer.

Note. We used a simple version of a family of procedures called the *rule of false position* or *regula falsi*. It is very different from the usual "guess and test," which depends on the fact that answers are often engineered to be small integers. By contrast, *regula falsi* led us, in one step after the "guess," to the answer. Fairly elaborate variants of the method were taught as far back as ancient Egypt. They were also taught very early in China, under names such as *ying pu tsu* (too much and not enough), in India. and in the Islamic world. *Regula falsi* was also a standard technique in Europe, finally—and regrettably—disappearing from the schools sometime in the late nineteenth century.

Another way: We could use algebra. Let n be the number of people. Then the number of dishes of rice is $n/3$, the number of dishes of vegetables is $n/4$, the number of dishes of duck is $n/6$, as is the number of dishes of fish, and we conclude that

$$\frac{n}{3} + \frac{n}{4} + \frac{2n}{6} = 88.$$

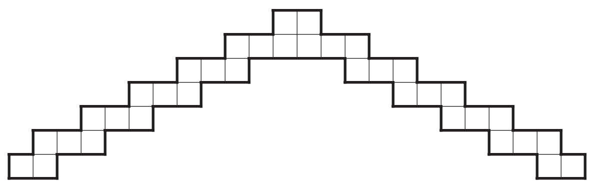
Multiply by 12 to clear denominators. We get $11n = (12)(88)$, and conclude that $n = (12)(88)/11 = 96$. Or if we want to be more complicated still, observe that

$$\frac{n}{3} + \frac{n}{4} + \frac{2n}{6} = n\left(\frac{2}{3} + \frac{1}{4}\right) = \frac{11n}{12} = 88$$

and solve for n.

Note the skill with fractions, and the higher level of abstraction, that the algebraic approach requires. *Regula falsi* is far more concrete.

2. The figure below was put together from thirty-six little squares. The area of the figure is 100 square centimetres. What is the perimeter of the figure?



Solution. The area is 100 and the figure was assembled from 36 identical squares—we could count them, they are outlined in the picture. So each square has area 100/36. It follows that each square has side $\sqrt{100/36}$, that is, 5/3.

Now we go around the zig-zag figure and count. With a little care we find that the outside of the figure is made up of 72 segments of length 5/3, for a perimeter of 120.

Comment: Counting to 72 isn't hard, but 72 is large enough that it is possible to miscount. Things get easier if we notice the left-right symmetry. So start at the top middle point of the figure, continue (say counterclockwise) until you get to the bottom middle point, and double.

Whether or not we take advantage of the mirror symmetry, it is worthwhile to compute *separately* the parts of the perimeter made up of horizontal line segments, and the parts made up of vertical line segments. For one thing, it is annoying and distracting to keep changing direction.

But more importantly, as we go around the boundary of the figure, we will go east just as much as we go east, and south just as much as we go north. So we can just look at our westward motion and our southward motion, and double. (If we are working with the left half only, the idea works fine for east and west, but needs to be modified slightly for north and south.) Instead of counting to 72, we end up countinng to 18. Much easier!

Here is another frequently helpful idea. The large "bay" at the bottom makes things harder—let's make it go away. Place an east-west line snug against the bottom of the country. Think of the line as a mirror, and reflect in this mirror the part of the shoreline that goes "in." (A picture would take too much space.)

We get a new country, with a much larger area, but with the *same* perimeter. This perimeter is easy to find! For it is the same as the perimeter of a *rectangle* that has the same north-south and the same east-west dimensions as the new country. This rectangle has base 24 sides of the little squares, and height 12 sides of the little squares. The little squares are $\frac{5}{3} \times \frac{5}{3}$, so the perimeter of the rectangle is 120.

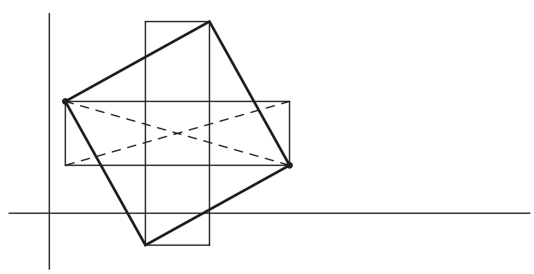
3. The vertices of square, taken counterclockwise, are $A(1, 7)$, $B(s, t)$, and $C(15, 3)$, and D. Find (s, t).

Solution. If we make a careful drawing on graph paper, the answer leaps out. Let's make a not so careful drawing and think about it. In going from A to B, we go right by a certain amount p, and down by a certain amount q. In symbols, $s = 1 + p$, $t = 7 - q$. In going from B to C we go up by the amount q, and right by the amount p. In symbols, $15 = s + q$, $3 = t + p$. We have the four simple linear equations

$$s = 1 + p, \quad t = 7 - q, \quad 15 = s + q, \quad 3 = t + p.$$

From the first and fourth equation, we get $s + t = 4$. From the middle two, we get $s - t = 8$. Finally, solve for s and t. We get $s = 6$, $t = -2$.

Another way: We can do the same thing without algebra. Plot the points A and C, and draw a rectangle as follows. Go straight down from A until you get to a point which is level with C (has the same y-coordinate as C), left until you get to C, up from C until you are at the same level as A, then left back to A.



Now draw a rectangle that forms a symmetrical "cross" with the first rectangle. The actual construction can be done in several ways, though it will turn out that it is enough to *think* about the construction.

The width and and height of the first rectangle can be found by counting squares, or by subtraction. Locate its center by drawing the diagonals. The second rectangle is identical to the first with directions reversed. We can draw it by counting squares from the center. Once the cross has been constructed, join points on it as shown. Symmetry shows that we obtain the desired square.

In our case, the first rectangle has base 14 and height 4, for a difference of 10. This means that the 4 pieces of the cross that stick out beyond the central square are all $(4 \times 5$. So to get to B from A, we go right 5 and down 9. The result is $(6, -2)$.

Another way: The square has central symmetry. We exploit the symmetry by finding the center. (The circle has greater central symmetry—the center is useful in almost any problem about circles.) The point halfway between A and B has coordinates $(8, 5)$. We get to A from the center by going left 7 and down 2. Thus we get to B from the center by going left 2 and down 7. It follows that $B = (6, -2)$.

Another way: The same idea can be carried out by first dragging $(8, 5)$ to the origin, where centers ought to be, by pulling it left by 8 and down by 5. Of course we have to shift the other points in the same way. Call the shifted points A', B', C', and D'.

The point A' has coordinates $(-7, 2)$, while C' has coordinates $(7, -2)$. The answer now jumps out: C' has coordinates $(-2, -7)$ (and D' has coordinates $(2, 7)$).

Push the points back to their original positions, by adding 8 to the first coordinate and 5 to the second. We find that the coordinates of B are $(6, -2)$.

Comment: We have used a simple instance of a powerful technique sometimes called "Transform, Solve, Transform Back."

Here is a more interesting geometric example. We want to find the point P on the top half of the ellipse $x^2 + 4y^2 = 1$ such that the tangent line to the ellipse at P passes through $(12, 0)$. Double y coordinates. The ellipse is transformed into the unit circle. The point P is moved to a point P' whose y coordinate is twice that of P. The tangent line to the ellipse at P is transformed into the tangent line to the circle at P', and $(12, 0)$ stays fixed.

We can now find P' by using standard properties of circles, for example the fact that the tangent line at P' is perpendicular to the radius through P'. And once we have P', we transform back by halving the y-coordinate.

The "Transform, Solve, Transform Back" idea is used throughout Mathematics, both elementary and advanced. Most "substitutions" and much of Linear Algebra can be interpreted in this way.

Another way: We can use more machinery. Note that B is equidistant from A and C. This yields the equation

$$(s - 1)^2 + (t - 7)^2 = (s - 15)^2 + (t - 3)^2,$$

which simplifies to the linear equation $28s - 8t = 184$. (The same equation can be obtained from the fact that the line joining B to the center is perpendicular to the line AC.)

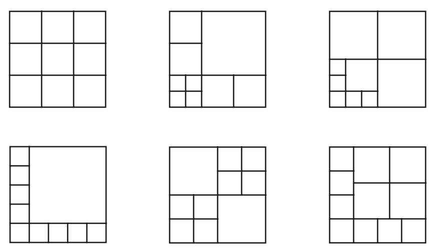
The lines AB and BC are perpendicular, so the product of their slopes is equal to -1. This yields the equation

$$(t-7)(t-3) = -(s-1)(s-15).$$

Use the linear equation to solve for t in terms of s, and substitute for t in the quadratic above. We get a quadratic equation in s, which we then solve. Messy!

4. Show how to cut up a square into (i) 9 squares; (ii) 10 squares; (iii) 11 squares; (iv) 2005 squares.

Solution. There is an obvious way to split a square into 9 squares, and there are less obvious ways—please see the top 3 squares.



The squares on the bottom left and bottom center are split into 10 squares. The square at the bottom right is split into 11 squares.

Now turn to 2005. Maybe we should look at the general problem of splitting into n squares.

The square on the bottom left has been split into an L-shaped collection of 9 squares on the west and south, together with a single square to the north-east. Imagine that the square we are splitting is 1×1, and let $k \geq 2$ be an integer. Imagine putting an "L" of width $1/k$ on the west and south, and dividing it in the natural way into $2k-1$ squares. The picture on the bottom left illustrates the case $k = 5$.

The $2k-1$ "small" squares together with the square on the north-east give us a decomposition of the original square into $2k$ squares. So we have an easy way to cut a square into any *even* number of squares greater than 2.

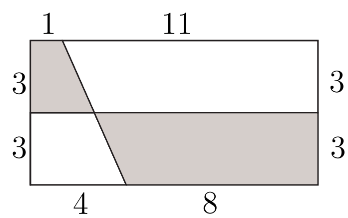
What about *odd* numbers of squares? Let n be any odd number greater than 5. Then $n-3$ is an even number greater than 2, so using the "L" method we can divide the original square into $n-3$ squares. Take any of these smaller squares and split it into 4 squares. We gain 4 squares but lose 1, leaving a total of n. The decomposition into 11 squares (bottom right) was done in this way.

Thus a square can be cut into any number $n \geq 6$ of squares. Unless n is small, there are many ways of doing it. A square can also be cut into 4 squares, and "cut" into 1 square. A square cannot be cut into 2, 3, or 5 squares.

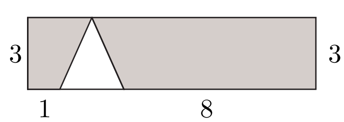
Another way: There is a simpler way to deal with 2005. Start with 1 square and split it into 4 squares. Then take one of the little squares and split it into 4 squares. We now have 7 squares. Take one of these squares and split it into 4 squares. We now have 10 squares. We can continue in this way, splitting the original square into 13 squares, 16, 19, and so on. What kind of numbers are we getting? All the numbers that leave a remainder of 1 on division by 3. But 2005 is such a number.

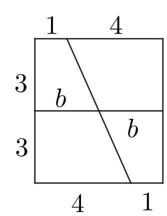
Comment: Similar ideas can be used to divide a cube into 2005 cubes. When we divide a cube into 8 cubes, we lose 1 cube and gain 8 for a net gain of 7. Since $2005 = (7)(286) + 3$, all we need to do is to find a division into n cubes where $n < 2005$ and n leaves a remainder of 3 on division by 7. After we have such an n, repeated splitting of cubes into 8 cubes will get us to 2005. Finding such an n isn't hard. For example, divide the cube into 27 cubes, and divide 5 of the little cubes into 27 cubes. We end up with 157 cubes, and $157 = (7)(22) + 3$.

5. Find the shaded area.



Solution. Reflect the top left part in the center line, and throw away the top half of the figure.





The area shaded in the new picture is the same as the area shaded in the old one. But in the new picture the shaded region is a 12×3 rectangle with the white triangle removed. The white triangle has base 3 and height 3, so area $9/2$. The shaded area is therefore $36 - 9/2$.

Another way: The shaded area is made up of two trapezoids, one on the upper left and the other on the lower right. *If* we know how to compute the area of a trapezoid, or are willing to discover how, we should be able to find the shaded area. There is one small problem: we do not know the base of the trapezoid on the upper left. We could find it, but it turns out we don't need to.

Let x be the length of the base of the upper left trapezoid. Then the two parallel sides of this trapezoid have length 1 and x, so the area of the trapezoid is $(1/2)(1+x)(3)$.

The two parallel sides of the lower right trapezoid have length $12-x$ and 8, and therefore the area of this trapezoid is $(1/2)(20-x)(3)$. Add up. The shaded area A is given by

$$A = \frac{(1+x)(3)}{2} + \frac{(20-x)(3)}{2} = \frac{63}{2}.$$

Another way: The algebraic cancellation has a natural geometric interpretation. In the picture of the problem, take the upper left trapezoid, turn it through 180°, and place it at the right of and flush against the lower right trapezoid (picture omitted). We conclude that the old shaded region has the same area as that of a trapezoid of height 3 whose two parallel sides have length 12 (upper) and 9 (lower). The area of this trapezoid can be found in various ways.

Another way: It is in fact easy to find x. We can use analytic geometry. Take the origin at say the bottom left-hand corner, and let the positive x-axis run along the bottom of the rectangle. Then the slanted line passes through $(4,0)$ and $(1,6)$, so it has equation $y=(-2)(x-4)$. Put $y=3$. We conclude that $x=5/2$

There are also simple geometric approaches. In the original picture, just keep the stuff to the left of the slanted line, duplicate it, turn it through 180° and place it as on the figure on the right. We get a 5 rectangle. By symmetry, $x=5/2$. Now that we know x, there are several ways to calculate the shaded area.

Comment: The original rectangle was divided into two halves by the horizontal line, in order to make possible many approaches to the problem. If the horizontal line splits the rectangle into unequal parts, then it is probably easiest to compute what we have called x. Instead of being an average, x is now a weighted average. If instead of being parallel to the base, the "midline" is slanted, the algebraic approach still works in more or less the same way.

It is said that when an Eygyptian king (one of the Ptolemies) asked Euclid whether there was an easier way to approach the theorems of geometry, Euclid replied "Sire, there is no royal road to geometry." The story is presumably false, for it is also told about Alexander the Great and his tutor, the minor mathematician Menaechmus. Dissing a king can be dangerous

Alan	Beti	Gamal
1	1	9
1	2	8
1	3	7
1	4	6

to your health, or at least raise problems at grant renewal time. Anyway, the story is too instructive to be true.

And in fact a royal road *is* provided by analytic geometry, as pioneered by Fermat and Descartes: coordinatize, express the geometric problem in terms of equations, and solve these equations. The process can (in principle) be automated, so a king with a large enough computing budget could prove all of Euclid's theorems.

6. In how many different ways can 11 identical muffins be distributed among Alan, Beti, and Gamal if each must receive at least one muffin?

Solution. List, or begin to list, all the ways that the muffins can be distributed. It turns out that there are 45, quite a few. Students to whom the problem appears difficult can be encouraged to look at a smaller problem, say with 7 muffins.

There are many ways to make the list systematic. A natural approach is to list all the ways in which Alan gets 1 muffin, then all the ways in which Alan gets 2 muffins, and so on up to all the ways in which Alan gets 9 muffins. Note that we are listing Beti's possible take in increasing order. It is clear that Beti can get anything from 1 to 9 muffins, with Gamal getting the rest. So Alan gets 1 muffin in 9 entries of our table.

Now start listing the ways in which Alan gets 2 muffins. Then Beti can get 1, 2, 3, $\ldots 8$, with Gamal getting the rest. This part of the table has 8 entries.

It is time to stop listing, and to start to *imagine* listing. There are 7 ways in which Alan gets 3 muffins, 6 in which he gets 4, and so on, up to the 1 way in which he gets 9. It follows that the total number of ways is

$$9 + 8 + 7 + 6 + \cdots + 3 + 2 + 1.$$

Add up. We get 45.

Another way: Imagine the muffins laid out in a row, like this:

$$\mathcal{M} \quad \mathcal{M} \quad \mathcal{M} \quad \mathcal{M} \quad \mathcal{M} \quad \mathcal{M} \quad \mathcal{M} \quad \mathcal{M} \quad \mathcal{M} \quad \mathcal{M} \quad \mathcal{M}$$

There are 10 *gaps* in this line of 11 muffins. We will decide how many muffins Alan, Beti, and Gamal get by first *choosing* two of these gaps to put dividing lines into, maybe like this:

$$\mathcal{M}\quad \mathcal{M}\quad \mathcal{M}\quad \mathcal{M}\Big|\quad \mathcal{M}\Big|\quad \mathcal{M}\quad \mathcal{M}\quad \mathcal{M}\quad \mathcal{M}\quad \mathcal{M}\quad \mathcal{M}$$

We give everything up to the first divider to Alan, then everything up to the next divider to Beti, and the rest to Gamal. It is clear that to every way of distributing the muffins there is exactly one way of putting up dividers.

So the number of ways of distributing the 11 muffins is the same as the number of ways of *choosing* 2 gaps from the 10. This number is variously called $\binom{10}{2}$, ${}_{10}C_2$, $C(10,2)$. Surprisingly many grade 7 students know how to compute the number of games in a round robin tournament that involves 10 teams, which is simply $\binom{10}{2}$.

The idea generalizes. We want to distribute n muffins among r people so that everyone gets at least one. Line up the muffins like above. That leaves $n-1$ gaps. We want to *choose* $r-1$ of these gaps to put dividers into. The number of ways of doing this is $\binom{n-1}{r-1}$.

Comment: It is not hard to explain to students how to calculate say $\binom{10}{2}$. There are 10 people in a room, and everybody shakes hands with everybody else. How many handshakes are there? We give two approaches to the problem, the first familiar to many students, and the other less familiar.

Call the people A_1, A_2, A_3, …, A_{10}. Count first the handshakes that A_1 is involved in: clearly there are 9. Now count the ones that A_2 is involved in, and that we have *not* already counted. There are 8. Now count the ones that A_3 is involved in and that we have not already counted. There are 7. Go on in this way: the total number is $9+8+\cdots+2+1$.

We count the same thing in a more useful way. Imagine that each person keeps a journal, and puts a ✓ in her journal for every hand she shakes. Obviously A_1 ends up with 9 ✓ in her journal, and so do A_2, A_3, …, A_{10}. Thus the total number of ✓ is (10)(9). Is this the number of handshakes? No, because any handshake resulted in *two* ✓. It follows that the number of handshakes is (10)(9)/2.

Precisely the same argument shows that

$$\binom{n}{2} = 1+2+3+\cdots+(n-1) = \frac{n(n-1)}{2}.$$

An interesting variant of the muffin problem is to ask for the number of ways of distributing the muffins so that everyone gets 0 or more muffins. We can go through a listing argument that is much the same as the one given above. Or else we can use the "divider" approach. To distribute n muffins among r people so that everyone gets 0 or more, first distribute $n+r$ among them so that everyone gets 1 or more. The divider argument shows there are $\binom{n+r-1}{r}$ ways of doing this. Now take away a muffin from everyone.

7. Alva left all her money to her three grandchildren, Beti, Cecil, and Delbert. Beti got half the money, plus \$1000. Cecil got half of what was left after that, plus \$2000. And Delbert got the remaining \$5000. What was the total amount of money Alva left?

Solution. It is natural to slip into the "algebra" approach. Let a be the amount of money that A left. Then B got $a/2+1000$. That left $a-(a/2+1000)$, that is, $a/2-1000$, for C and D.

C got half of that plus 2000, so C got $a/4-500+2000$, that is, $a/4+1500$. That left $(a/2-1000)-(a/4+1500)$, that is, $a/4-2500$ for D. Thus $a/4-2500=5000$, and therefore $a=30000$.

If we use this approach, it may be better to note in advance that we will be dividing by 2 a couple of times. So it is probably a good idea to let the amount that A left be $4x$. The calculations are much the same, but fraction-free.

Another way: Think of the distribution of the bequest as taking place in stages as follows: (i) B gets half the money; (ii) B gets 1000; (iii) C gets half of what's left; (iv) C gets 2000; (v) D gets 5000.

Now *work backwards* from the 5000. We get 7000, 14000, 15000, 30000.

Another way: We can "guess and test." This works quite well for the numbers in our problem, and less well if Delbert gets \$4327.98.

Guess for example that the bequest was \$20000. Then B gets 11000. This leaves \$9000, which we can already see is too little. Let's persist anyway. We find that C gets \$6500, leaving \$2500 for D.

But D should get \$5000, so maybe we should guess that the bequest was \$40000. With \$40000, we have \$19000 after B gets her share, so \$11500 for C and therefore \$7500 for D.

OK, \$40000 is too much. We can make a guess between 20000 and 40000, decide whether it is too much or not enough, and continue. It will not take long to get to the correct answer of 30000.

Note that 5000 is halfway between 2500 and 7500, while 30000 is halfway between 20000 and 40000. This is not an accident: what D gets is a *linear* function of the bequest, and therefore the bequest is a linear function of what D gets.

More informally, imagine *increasing* the bequest by 1 dollar. Then B gets 50 cents of the increase, with the rest shared equally by C and D. So to change D's take by 1 dollar, we must change the bequest by 4 dollars.

Now we can make a *single* "guess," and then immediately jump to the right answer. Take our earlier guess that the bequest is \$20000. We found that D got \$2500, which is \$2500 short of the truth. So we must get an additional \$2500 to trickle down to D. Since only 1 out of every 4 dollars trickles down, we need to add 4 times 2500 to the 10000, for a total of 30000.

8. How many ways are there to label the six faces of a cube with the labels 1, 2, 3, 4, 5, 6 ? Two labellings are different if one can't be obtained from the other by a rotation of the cube. (Real dice have their faces labelled so that the numbers on opposite faces always add up to 7, but we are not making that restriction here.)

Solution. Let's put the cube on a table with the "1" face down. There are 5 possibilities for the top face. We will count how many different labellings there are with the top face showing a 2. There will be just as many labellings with top face labelled 3, 4, 5, or 6. So to find the number of labellings, we count the number that have 1 on the table and 2 on top, and multiply by 5.

With the 1 staying in contact with the table, rotate the cube until the face labelled 3 is facing you. Then there are 3 possibilities for the label to the right of the "3" face:: 4, 5, or 6. We count the number of possibilities with 4 on the right. It is clear there are two (5 or 6 on the back face). Similarly, there are 2 possibilities with 5 on the right of the 3 face, and 2 possibilities with 6 on the right of the 3 face, for a total of 3×2.

Now that we have counted the number of possibilities with 1 on bottom and 2 on top, multiply by 5 as explained in the first paragraph: the number of labellings is $5 \times 3 \times 2$.

9. There are altogether 15 positive integers that divide 400. Find the product of these 15 numbers.

Solution. The number 400 doesn't have many factors, so we can list them all and multiply. Things should go OK even if the listing is not systematic. The risk of leaving out factors is small, since the number of factors has been announced in the problem.

Systematic is better. Note for example that if $ab = n$, then each of a and b, that is, a and n/a, is a factor of n. It is convenient to list the factors of 400 in such pairs a, n/a, for example as

$$1, 400; \quad 2, 200; \quad 4, 100; \quad 8, 50; \quad 16, 25; and finally, 20.$$

The product of the two numbers in a pair is 400, and there are 7 pairs, together with an unpaired 20, so the product of all the factors is $(400^7)(20)$. We now have our answer, but we may wish to multiply out. The multiplication is easy: we get 327680000000000000000. Actually, unless the "long" answer is really needed, it is better not to multiply. The expression $(20^7)(20)$ is more compact and more informative.

Comment: Look at the problem more generally. We want to find the product of all the factors of n. Look first at numbers n like 200 which are *not* perfect squares. Then the factors of n come in pairs. Suppose there are d factors in all. Then there are $d/2$ pairs, and each pair has product n. So the product of all the factors is $n^{d/2}$.

Look next at the case when n is a perfect square, like 400, and suppose again that n has d factors. The factors come in $(d-1)/2$ pairs, plus the unpaired factor $\sqrt{n}$, which it is more pleasant to call $n^{1/2}$. The product of all the factors is therefore $(n^{(d-1)/2})(n^{1/2})$. This turns out to be $n^{d/2}$, so actually we can use the same formula whether or not n is a perfect square.

Comment: If we know the prime factorization of n, we can calculate the number d of divisors of n without making an explicit list. We illustrate with a particular n, but the idea is general. Let n have the prime factorization

$$n = 3^4 \times 11^6 \times 31^1.$$

Any (positive) divisor k of n has the shape

$$k = 3^a \times 11^b \times 7^c,$$

where $a = 0$, 1, 2, 3, or 4, $b = 0$, 1, 2, 3, 4, 5, or 6 and $c = 0$ or 1.

To pick a divisor k of n, we first decide to what power we will take the prime 3. There are clearly 5 choices, namely 0, 1, . . . , 4. For *every one* of these choices, there are 7 choices for the power to which we will take the prime 11, namely 0, 1, . . . , 6. So there are already 5×7 ways of deciding the power to which 3 occurs in k *and* the power to which 11 occurs in k. Finally, decide to what power 31 will occur. For every decision about 3 and 11, there are 2 decisions about 31. So the total number of possible decisions is $5 \times 7 \times 2$.

10. Candles A and B each burn at a uniform rate. But because A is thicker than B, it burns down more slowly. The candles were lit at 7:00. At 8:00, the candles were of the same height. At 11:00, candle B was finished. And at 12:00, so was A. At 7:00, A was 18 cm high. How high was B?

Solution. We will approach the problem algebraically, trying to keep things reasonably simple. We should probably measure time from 7:00, so 7:00 is time 0, 8:00 is time 1, 11:00 is time 4, and 12:00 is time 5.

Let b be the initial height of B, let r be the burning rate of B in centimetres per hour. Since B died at time $t = 4$, we have $b = 4r$. The burning rate of A is clearly 18/5 cm per hour.

At $t = 1$, the candles were the same height. Clearly A had height $18-18/5$, and B had height $b - r$, and therefore

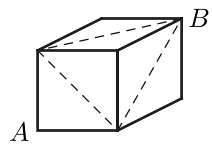
$$b - r = 18 - \frac{18}{5} = \frac{72}{5}.$$

But $b = 4r$. It follows that $3r = 72/5$, and now a little calculation gives $r = 24/5$ and therefore $b = 96/5 = 19.2$.

Another way: We can avoid the variables, but it takes some concentration. At 8:00 the candles were the same height, namely $18-18/5$, or $72/5$. From that time, it took 4 hours for A to die, and 3 for B. So for every 3 cm that A loses, B loses 4. Thus for every cm that A loses, B loses $4/3$. Now work backwards from 8:00. From 7:00 to 8:00, A lost $18/5$, so B lost $(18/5)(4/3)$, that is, $24/5$. It follows that at 7:00 the height of B was

$$\frac{72}{5} + \frac{24}{5}.$$

11. The picture is of a box of the usual shape. Three face diagonals are shown; they have lengths 39, 40, and 41. Find the distance from A to B.



Solution. First we find an expression for the distance between A and B in terms of the sides of the box. Imagine heading straight up from A until we get to the dashed line that goes towards B. Let C be the point that we reach.

Let w be the height of the box, that is, the distance between A and C, and let the other two dimensions of the box be u and v.

By the Pythagorean Theorem, $(BC)^2 = u^2 + v^2$. But $\triangle ACB$ is right-angled at C. Therefore, by the Pythagorean Theorem again,

$$(AB)^2 = (BC)^2 + (AC)^2 = u^2 + v^2 + w^2.$$

This expression for the length of the "long diagonal" is pleasantly simple. Of course it is symmetrical in u, v, and w, since it is geometrically obvious that the four long diagonals are equal. The argument we used broke symmetry. A symmetrical justification would be nice.

Now we turn to our problem. Let x, y, and z be the lengths of the sides of the box. If they are listed in increasing order, we have

$$x^2 + y^2 = 39^2, \quad x^2 + z^2 = 40^2. \quad y^2 + z^2 = 41^2.$$

This is a system of linear equations in x^2, y^2, and z^2, so we could solve for these, and then compute $x^2 + y^2 + z^2$. But there is a more symmetrical way of doing things: just *add* the three equations.

By the way, the expression "add equations" does not make logical sense. An equation is an *assertion* that two things are equal. We can add *quantities,* not assertions. Unfortunately, however, "add equations" is a standard part of school language, and it is impossible to eradicate it. It is not a big problem, everybody—maybe—knows what is meant.

We arrive at

$$2(x^2 + y^2 + z^2) = 39^2 + 40^2 + 41^2.$$

Now just bring out the calculator. But mine isn't working. So I will note that the right-hand side of the above equation is equal to

$$(40 - 1)^2 + 40^2 + (40 + 1)^2.$$

Imagine expanding the two ends. Note that the "middle" terms cancel, so our sum is equal to $3(40^2) + 2$. Now we can do the arithmetic in our heads. It turns out that $x^2 + y^2 + z^2 = 2401$. So $AB = \sqrt{2401} = 49$.

12. Twenty percent of the people who like chocolate like hot pepper. Ninety percent of the people who like hot pepper like chocolate. Everyone likes one or the other or both. What fraction of the people like both?

Solution. A picture ("Venn" diagram) is useful to essential here, so that one can concentrate on the tripartite division of the world into chocolate-only, pepper-only, and chocolate-pepper. By the way, "Venn" diagrams were used by Euler long before Venn was born.

Decide, at the beginning quite arbitrarily and possibly wrongly, on how many people there are in one of the relevant categories. Maybe because chocolate was mentioned first, let's see what happens if we have 100 people who like chocolate. Then 20 of them like hot pepper. These 20 represent 90% of the people who like hot pepper. But 20 is not 90% of any integer: in order to be 90% of an integer, a number has to be divisible by 9. So if we start with a number C of people who like chocolate, we want C to be divisible by 5 (so that we can take 20% of it) and also divisible by 9 (so that $C/5$ will be a multiple of 9). That suggests taking $C = 45$. Certainly C must be a multiple of 45.

If $C = 45$, then we get 9 people who like both. And 9 is 90% of the people who like hot pepper, so in total there are 10 who like pepper. The number of people in the group is then $(45 + 10) - 9$, because the 45 and the 10 both include the people who like both. Thus there are 46 people in the group, of whom 9 like both, for a fraction of 9/46.

Would the answer be different if we had a number of people other than 46 in the group? Probably not, since the wording of the problem encourages us to take it for granted that the number of people is irrelevant. But it might be worthwhile to repeat the argument with $45N$ people who like chocolate. The calculations are almost identical, and the N cancels.

Another way: Problems of this type can be confusing, and "algebra" can be useful as an accounting device, to keep things clear. Maybe also for the sake of symmetry, or variety, we should start with the number of people who like both, and call it x.

Then x is 20% of the people who like chocolate. It follows that there are $(100/20)x$ people who like chocolate. Similarly, there are $(100/90)x$ people who like hot pepper. The total number of people in the group is therefore

$$(100/20)x + (100/90)x - x,$$

since the sum of the first two terms double counts the people who like both. Thus the fraction who like both is

$$\frac{x}{(100/20)x + (100/90)x - x}.$$

Now it's all over. The x, not surprisingly, cancels, and a bit of manipulation of fractions yields $9/46$.

13. Given that a is a number such that $\left|a - \frac{1}{a}\right| = 1$, what can we conclude about $\left|a + \frac{1}{a}\right|$?

Solution. Note that

$$\begin{aligned} \left(a - \frac{1}{a}\right)^2 &= a^2 - 2 + \frac{1}{a^2} \qquad \text{and} \\ \left(a + \frac{1}{a}\right)^2 &= a^2 + 2 + \frac{1}{a^2} \qquad \text{and therefore} \\ \left(a + \frac{1}{a}\right)^2 &= \left(a - \frac{1}{a}\right)^2 + 4 = 1 + 4 = 5. \end{aligned}$$

Finally, use the fact that use the fact that in general $|u| = \sqrt{u^2}$ to conclude that $a + 1/a| = \sqrt{5}$.

Another way: There are more awkward ways to proceed. We could solve the equation

$$\left|a - \frac{1}{a}\right| = 1, \quad \text{or equivalently} \quad a - \frac{1}{a} = \pm 1.$$

We can find the four solutions of these equations, and for every solution a compute $|a + 1/a|$. It is not quite as bad as it sounds. For a is a solution of $a - 1/a = 1$ if and only if $-a$ is a solution of $a - 1/a = -1$, so we really only need to solve one equation.

Look now at $a - 1/a = 1$. This is equivalent to $a^2 - a - 1 = 0$, whose solutions are $(1 \pm \sqrt{5})/2$. Thus

$$a + \frac{1}{a} = \frac{1 \pm \sqrt{5}}{2} + \frac{2}{1 \pm \sqrt{5}}.$$

Inserting absolute value signs gives us an answer. But it is worth trying to simplify, for example by "rationalizing the denominator." Fairly quickly we reach $\sqrt{5}$.

14. Let $\mathcal{A}$ be the set of all points (x, y) such that $|x + y| + |x - y| \leq 4$. Find the area of $\mathcal{A}$.

Solution. Look first for obvious symmetries. If (a, b) satisfies the inequality, then so does $(-a, -b)$. We obtain $(-a, -b)$ by rotating (a, b) about the origin through $180°$, or by reflecting (a, b) in a point mirror at the origin, or by reflecting (a, b) in the x-axis, then reflecting the result in the y-axis.

However we think of things, we have cut the work down by half. If, for example, we find out what the part of our region to the right of the y-axis looks like, the rest can be filled in mechanically.

Absolute values can be difficult to work with, so it would be nice to throw away the absolute value signs. We will do exactly that, carefully.

The following description of the absolute value function out to be useful.

$$|u| = \begin{cases} u, & \text{if } u \geq 0; \\ -u & \text{if } u < 0; \end{cases}$$

In formal treatments, the equations above are often used as the *definition* of the absolute value function. They are not needed when we deal with specific numbers, but are handy when we deal with algebraic expressions.

For comfort, look first at stuff in the first quadrant. So $|x+y|$ is no trouble, it is just $x + y$. And $|x - y| = x - y$ if $x \geq y$. It may be more familiar to rewrite $x \geq y$ as $y \leq x$. We have $y = x$ on the familiar line, and $y \leq x$ on or below that line.

So in the first quadrant, and on or below $y = x$, we have $|x+y|+|x-y| = (x + y) + (x - y) = 2x$, and therefore our inequality holds if $2x \leq 4$, that is, if $x \leq 2$. The region we have identified can more simply be described as the triangle with corners $(0, 0)$, $(2, 0)$, and $(2, 2)$.

Now look at the part of the first quadrant that has $x \leq y$. This is easy to deal with. For note that $|x-y| = |y-x|$, and therefore in the first quadrant and above the line $y = x$ we are looking at the inequality $(x+y)+(y-x) \leq 4$, that is, $y \leq 2$. This gives us the triangle with vertices $(0, 0)$, $(2, 2)$, and $(0, 2)$. Another way of saying the same thing is that since $|x-y| = |y-x|$, for the part of the first quadrant with $y \geq x$, we have the situation of the preceding paragraph with x and y interchanged, so we just reflect in the line $y = x$.

Our two triangles together make up a 2×2 square with one corner at the origin, and two sides along the axes.

Now go on to the fourth quadrant. Here y is negative. Temporarily, let $y = -y'$. So we are looking at the inequality $|x - y'| + |x + y'| \leq 4$, and trying to identify non-negative x and y' that work. But we had done this! The result is a square in the first quadrant of the $x - y'$ plane. Reflect across the x-axis to get back to the original problem. So the part of the region in the fourth quadrant is just obtained by reflecting the first quadrant part in $x = 0$.

Finally, fill in the rest by symmetry. We end up with four 2×2 squares (indeed a single 4×4 square, with area 16.

Another way: Let's look at the simpler problem $|u| + |v| \leq 4$. When u and v are non-negative, we are looking at $u + v \leq 4$, which is easy to identify: it is the region in the uv-plane on the boundary of or in the triangle with corners $(0, 0)$, $(4, 0)$, and $(0, 4)$. And since absolute value is not sensitive to sign, we get the parts in the other quadrants by reflection.

Our general strategy is to *transform* our original problem into the problem $|u| + |v| \leq 4$, solve the simpler problem, and then *transform back*.

Map any point (x, y) to (u, v), where $u = x + y$ and $v = x - y$. The problem $|x + y| + |x - y| \leq 4$ becomes $|u| + |v| \leq 4$, and the region in the uv plane is the square with corners $(4, 0)$, $(0, 4)$, $(-4, 0)$, $(0, -4)$.

But from $u = x + y$, $v = x - y$ we obtain $x = (u + v)/2$, $y = (u - v)/2$ (this is the transform back phase). The square in the uv-plane with corners $(4, 0)$, $(0, 4)$, and so on becomes the region with corners $(2, 2)$, $(2, -2)$, $(-2, -2)$, and $(-2, 2)$.

Comment: The same "Transform, Solve, Transform Back" method will work just as well in similar situations, for example an inequality of the type

$$|a_1x + b_1y + c_1| + |a_2x + b_2y + c_2| \leq k$$

as long as the lines $a_1x + b_1y = 0$ and $a_2x + b_2y$ are not parallel. The basic geometry is more or less the same in all cases. The region is a certain parallelogram and its interior.

15. Find all triples (a, b, c) of integers, all greater than or equal to 3, such that

$$\frac{1}{a} - \frac{1}{b} + \frac{1}{c} = \frac{1}{2}.$$

Solution. Subtraction is a bit less intuitive than addition, so it is useful to rewrite the equation as

$$\frac{1}{a}+\frac{1}{c}=\frac{1}{b}+\frac{1}{2}.$$

Note the symmetry between a and c: if the triple (x, y, z) is a solution of our equation, then so is the triple (z, y, x).

We cannot have both a and c greater than 3. For if each is greater than or equal to 4, then the left-hand side is less than or equal to $1/2$, so cannot be equal to the right-hand side.

Let's assume that $a = 3$. We can then take care of the case $c = 3$ by symmetry.

So our equation reduces to $1/3 + 1/c = 1/b + 1/2$, or equivalently

$$\frac{1}{c}=\frac{1}{b}+\frac{1}{6}.$$

This forces $c < 6$. So we have very few cases to worry about.

Suppose $c = 3$. That gives $b = 6$, and we get the triple $(3, 6, 3)$.

Suppose $c = 4$. That gives $b = 12$, so we have the triple $(3, 6, 4)$ and by symmetry $(4, 6, 3)$.

Suppose $c = 5$. That gives $b = 30$, and we get the triples $(3, 30, 5)$ and by symmetry $(5, 30, 3)$.

Comment: The equation of this problem comes up in the analysis of regular polyhedra. The 5 solutions we have found correspond to the 5 regular polyhedra (tetrahedron, cube, octahedron, dodecahedron, and icosahedron).

16. A restaurant bought 48 kg of swordfish and 48 kg of tuna. If the restaurant had divided the money it spent on fish equally between swordfish and tuna, it would have been able to buy 2 more kg of fish. Given that a kilogram of swordfish costs $10.00, what can we conclude about the price of a kilogram of tuna?

Solution. It is more instructive to work with letters. Let s be the cost of a kg of swordfish, and t be the cost of a kg of tuna. Suppose that the restaurant bought k kilograms each of swordfish and tuna. Suppose also that the restaurant could have bought m more kilograms of fish if it had divided its money equally between the two kinds of fish.

We know s, k, and m. It would maybe feel a little easier to work with the particular numbers of the problem rather than with letters. But there are good reasons for working with letters rather than specific numbers. We get increased generality. Also, letters are often less complicated to deal with than numbers. And finally and most importantly, letters may reveal structure that numerical computation hides.

The actual cost of the fish was $k(s+t)$. If we spend half of this on swordfish and half on tuna, the amount of fish that we can buy is

$$\frac{k(s+t)}{2s} + \frac{k(s+t)}{2t},$$

(the first half of the expression is the swordfish term, and the second the tuna term). The amount above is m more than what we actually bought, which is $2k$. We obtain the equation

$$\frac{k(s+t)}{2s} + \frac{k(s+t)}{2t} = 2k + m.$$

Now we can substitute our known values of k, s, and m and solve for t. But it is more informative to keep working with letters.

Multiply both sides by $2st$. We obtain

$$k(s^2 + 2st + t^2) = (2k + m)(2st).$$

Rearrange, and divide through by k. We obtain the equation

$$s^2 - 2(1 + k/m)st + t^2 = 0.$$

It is maybe useful to let $1 + m/k$ be equal to a. We finally arrive at $s^2 - 2ast + t^2 = 0$.

At this stage, out of curiosity about the price of tuna we put in the specific numbers of the problem. The above equation is quadratic in t. Solve, using the quadratic formula. It turns out that $t = 4s/3$ or $t = 3s/4$, giving for tuna a price of 40/3 or 30/4 dollars per kilo.

Note that with our particular numbers the equation simplifies to $12s^2 - 25st + 12t^2 = 0$, which can be rewritten as $(4s - 3t)(3s - 4t) = 0$, so we did not need the quadratic formula. What does the 3–4–5 triangle have to do with the price of tuna?

Alternately, we could have let $x = t/s$. Then our equation would have simplified to

$$x + \frac{1}{x} = 2 + 2m/k = 2a,$$

nicely symmetrical, just like $s^2 - 2ast + t^2 = 0$.

Comment: It is obvious, at least in retrospect, that t is determined by the ratio m/k: the individual numbers m and k do not matter. The fact that the two ratios 4/3 and 3/4 have product 1 is not an accident. By symmetry, if r is a root of $x + 1/x = 2a$, then so is $1/r$.

17. Alphonse ran in a cross-country race, running half of the *distance* at 3 minutes per km and half at 3 minutes 10 seconds per km. If he had run half of the *time* at 3 minutes per km, and half at 3 minutes 10 seconds per km, it would have taken him 1 second less to finish the race. How long did Alphonse actually take?

Solution. We solve a somewhat more general problem. Let half the distance be run at a minutes per km, and half at b minutes per km. Suppose that running half the *time* at a and half the time at b would have saved m minutes. We will compute the time t it actually took to run the race. In our problem, we have $a = 3$, $b = 3 + \frac{10}{60}$, and $m = \frac{1}{60}$.

There are several reasons for working with a and b. We get increased generality. Also, letters are often less complicated to deal with than numbers. And finally and most importantly, letters may reveal structure that numerical computation hides.

It is reasonable to use "algebra." We denote certain obviously important quantities by letters, set up equations, and solve them.

Let half the distance be h. The time that Alphonse actually took was $ah + bh$, so $t = (a + b)h$.

If Alphonse had run half the time at a minutes per km, and half at b minutes per km, he would have saved m minutes. Half of what time? It would have taken him $t-m$ to finish the race, and half of that is $(t-m)/2$.

If we run for time $(t - m)/2$ at a minutes per km, we cover a distance $(t - m)/2a$. Similarly, if we run for time $(t - m)/2$ at b minutes per km, we cover a distance $(t - m)/2a$. The sum of these distances is the actual length of the race course, namely $2h$. We have obtained the equation

$$\frac{t-m}{2a} + \frac{t-m}{2b} = 2h = \frac{2t}{a+b}.$$

Solve for t. We get first

$$t\left(\frac{1}{2a} + \frac{1}{2b} - \frac{2}{a+b}\right) = m\left(\frac{1}{2a} + \frac{1}{2b}\right).$$

If we are working numerically, we are essentially finished, for we can calculate the coefficient of t, and the right-hand side, and divide. But since we are working with a and b, we will do some algebraic simplification first. Multiply through by $2ab(a + b)$. We obtain

$$t\left(b(a+b) + a(a+b) - 4ab\right) = m\left(b(a+b) + a(a+b)\right),$$

and after a little more work we get

$$t = m\left(\frac{a+b}{a-b}\right)^2.$$

Finally, put $m = 1/60$, $a = 3$, and $b = 3 + \frac{10}{60}$. It turns out that $t = 1369/60$, 22 minutes and 49 seconds.