UBC Workshops 2005
Solutions A

1. What time will it be 2005 hours from now?

Solution. There are—sort of—24 hours in a day. Divide 2005 by 24. The quotient is 83, the remainder is 13. So 2005 hours from now, the clock time will be the same as it will be 13 hours from now. For example, if it is now 10:30 am, it will be 11:30 pm.

Not quite! We have not taken account of what is called Daylight Saving Time in Canada, and Summer Time in England. If we live in Saskatoon, or Tokyo, or Nairobi, there is no Daylight Saving time, and hence no further problem.

There is no Daylight Saving Time in Saskatchewan, in the part of Arizona outside the Navajo Nation, and in most of Indiana. In the rest of North America, including Mexico, Daylight Saving Time starts at 2:00 am local time on the first Sunday in April, and ends at 2:00 am local time on the last Sunday in November. In the European Union, Summer Time starts at 1:00 am Greenwich Mean Time on the last Sunday in March, and ends at 1:00 am Greenwich Mean Time on the last Sunday in October. There are various rules in other parts of the world. Daylight Saving is largely absent in countries near the equator. Australia, New Zealand, and Chile are as usual upside down.

Suppose we are in Vancouver and it is now 8:30 am, February 15. Then 83 days carry us into Daylight Saving Time, so 2005 hours from now it will be 10:30 pm. If "now" is 10:30 am, September 15, we are in Daylight Saving Time. But 83 days from now it will be Standard Time, so it will be 10:30 pm. If "now" is for example in November or May, 83 days brings no time change, and we can pretend we are in Saskatoon.

2. Sylvia is strolling along Robson Street when someone grabs her purse and starts running away. The person is 25 metres ahead of her when Sylvia finally reacts and starts giving chase. Sylvia runs 5 metres for every 3 metres the thief runs. How many metres will Sylvia have to run before she catches up to the thief?

Solution. Sylvia gains 2 metres on the thief for every 5 metres that she runs, or 1 metre for every 2.5. She needs to gain 25 metres, so she must run (25)(2.5) metres, that is, 62.5 metres.

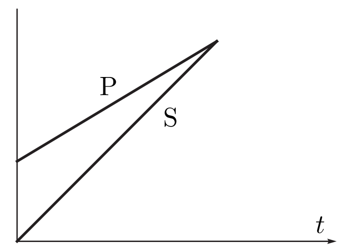
Another way: If we are in a more algebraic mood, and we shouldn't be, we can let s be the distance that Sylvia runs until she catches the thief. The distance the thief runs is $(3/5)s$. Since the thief had a 25 metre lead,

$$s = 25 + \frac{3s}{5}.$$

Standard manipulation yields $s = 125/2 = 62.5$.

Another way: We can "guess and check." As long as this is done with careful recording of results, it has value, and can lead to an answer. There may be a difficulty because students are accustomed to "guess and check" producing (small) integer answers.

Another way: Plot the space-time histories of our villain and hero, with $t = 0$ when Sylvia begins to give chase.



The line labelled P is the path in space-time of the purse after the snatching, and the line labelled S is Sylvia's path in space-time. It would be better to make the time axis vertical, but old habits are hard to break!

If things are carefully plotted, we should be able to measure the answer from the graph, or calculate the answer using a similar triangles argument. For the question asked here, this approach is overkill. But it connects in an interesting way two "different" areas of mathematics. Many simple rate problems can be approached graphically.

3. At a banquet, there was one serving dish of rice for every two people, one serving dish of meat for every three people, and one serving dish of fish for every four people. Altogether, 78 serving dishes of food were set out. How many people were at the banquet?

Solution. Banquet problems like this one have been part of the Chinese problem literature since -200, and may be much older.

We can "guess" that there were 12 people at the banquet. Of course the guess is too small. But it is convenient, since 12 is the smallest positive integer divisible by 2, 3, and 4.

Let's check whether the guess is right. If there really were12 people at the banquet, they had 6 serving dishes of rice, 4 of meat, and 3 of fish, for a total of 13. Wrong guess, there were actually 78 dishes of food served.

But note that $78/13 = 6$, so if we multiply the "guess" by 6, we get the right answer. We conclude that there were 72 people at the banquet. So by thinking about one guess that turned out to be wrong, we can go directly to the right answer.

Comment. We used a simple version of a family of procedures called the *rule of false position* or *regula falsi.* It is very different from the usual "guess and test," which depends on the fact that answers are engineered to be small integers. By contrast, *regula falsi* led us, in one step after the "guess," to the answer. Fairly elaborate variants of the method were taught as far back as ancient Egypt. They were also taught very early in China, under names such as *ying pu tsu* (too much and not enough), in India. and in the Islamic world. *Regula falsi* was also a standard technique in Europe, finally—and regrettably—disappearing from the schools sometime in the late nineteenth century.

Another way: We could use algebra. Let n be the number of people. Then the number of dishes of rice is $n/2$, the number of dishes of meat is $n/3$, the number of dishes of fish is $n/4$, and we conclude that

$$\frac{n}{2}+\frac{n}{3}+\frac{n}{4}=78.$$

Multiply by 12 to clear denominators. We get $13n = (12)(78)$, and conclude that $n = (12)(78)/13 = 6$. Or if we want to be more complicated still, observe that

$$\frac{n}{2}+\frac{n}{3}+\frac{n}{4}=n\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)=\frac{13n}{12}=78$$

and solve for n.

Note the skill with fractions, and the high level of abstraction, that the algebraic approach requires. *Regula falsi* is far more concrete.

4. Alva left all her money to her three grandchildren, Beti, Cecil, and Delbert. Beti got half the money, plus \$1000. Cecil got half of what was left after that, plus \$2000. And Delbert got the remaining \$5000. What was the total amount of money Alva left?

Solution. It is natural to slip into the "algebra" approach. Let a be the amount of money that A left. Then B got $a/2+1000$. That left $a-(a/2+1000)$, that is, $a/2-1000$, for C and D.

C got half of that plus 2000, so C got $a/4-500+2000$, that is, $a/4+1500$. That left $(a/2-1000)-(a/4+1500)$, that is, $a/4-2500$ for D. Thus $a/4-2500=5000$, and therefore $a=30000$.

If we use this approach, it may be better to note in advance that we will be dividing by 2 a couple of times. So it is probably a good idea to let the amount that A left be $4x$. The calculations are much the same, but fraction-free.

Another way: Think of the distribution of the bequest as taking place in stages as follows: (i) B gets half the money; (ii) B gets 1000; (iii) C gets half of what's left; (iv) C gets 2000; (v) D gets 5000.

Now *work backwards* from the 5000. We get 7000, 14000, 15000, 30000.

Another way: We can "guess and test." This works quite well for the numbers in our problem, and less well if Delbert gets \$4327.98.

Guess for example that the bequest was \$20000. Then B gets 11000. This leaves \$9000, which we can already see is too little. Let's persist anyway. We find that C gets \$6500, leaving \$2500 for D.

But D should get \$5000, so maybe we should guess that the bequest was \$40000. With \$40000, we have \$19000 after B gets her share, so \$11500 for C and therefore \$7500 for D.

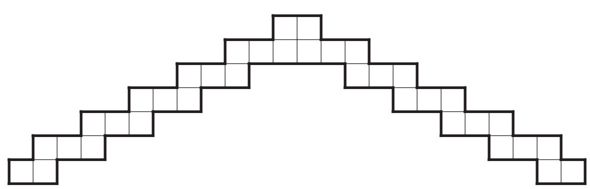
OK, \$40000 is too much. We can make a guess between 20000 and 40000, decide whether it is too much or not enough, and continue. It will not take long to get to the correct answer of 30000.

Note that 5000 is halfway between 2500 and 7500, while 30000 is halfway between 20000 and 40000. This is not an accident: what D gets is a *linear* function of the bequest, and therefore the bequest is a linear function of what D gets.

More informally, imagine *increasing* the bequest by 1 dollar. Then B gets 50 cents of the increase, with the rest shared equally by C and D. So to change D's take by 1 dollar, we must change the bequest by 4 dollars.

Now we can make a *single* "guess," and then immediately jump to the right answer. Take our earlier guess that the bequest is \$20000. We found that D got \$2500, which is \$2500 short of the truth. So we must get an additional \$2500 to trickle down to D. Since only 1 out of every 4 dollars trickles down, we need to add 4 times 2500 to the 10000, for a total of 30000.

5. The figure below was made by putting together 36 equal-sized squares. The area of the figure is 144 square centimetres. Find the perimeter of the figure.



Solution. The area is 144 and the figure was assembled from 36 identical squares—we could count them, they are outlined in the picture. So each square has area 144/36, that is, 4. It follows that each square has side $\sqrt{4}$, that is, 2.

Now we go around the zig-zag figure and count. With a little care we find that the outside of the figure is made up of 72 segments of length 2, for a perimeter of 144.

Comment: Counting to 72 isn't hard, but 72 is large enough that it is possible to miscount. Things get easier if we notice the left-right symmetry. So start at the top middle point of the figure, continue (say counterclockwise) until you get to the bottom middle point, and double.

Whether or not we take advantage of the mirror symmetry, it is worthwhile to compute *separately* the parts of the perimeter made up of horizontal line segments, and the parts made up of vertical line segments. For one thing, it is annoying and distracting to keep changing direction.

But more importantly, as we go around the boundary of the figure, we will go east just as much as we go east, and south just as much as we go north. So we can just look at our westward motion and our southward motion, and double. (If we are working with the left half only, the idea works fine for east and west, but needs to be modified slightly for north and south.) Instead of counting to 72, we end up counting to 18. Much easier!

Here is another frequently helpful idea. The large "bay" at the bottom makes things harder—let's make it go away. Place an east-west line snug against the bottom of the country. Think of the line as a mirror, and reflect in this mirror the part of the shoreline that goes "in." (A picture would take too much space.)

We get a new country, with a much larger area, but with the *same* perimeter. This perimeter is easy to find! For it is the same as the perimeter of a *rectangle* that has the same north-south and the same east-west dimensions as the new country. This rectangle has base 24 sides of the little squares, and height 12 sides of the little squares. The little squares are 2×2, so the perimeter of the rectangle is 144.

6. Chris has \$100 more than Eric. After Chris spends \$52, he has seven times as much as Eric. How much does Eric have?

Solution. To someone who has experience in translating problems into algebraic language, the problem is straightforward. Let x be the amount that Eric has. Then Chris started off with $x + 100$. After he spent \$52, he was left with $(x + 100) - 52$. This is equal to $7x$, and therefore

$$(x + 100) - 52 = 7x.$$

A little manipulation yields $6x = 48$. Thus $x = 8$.

Another way: The answer is a small integer, and can probably be reached quickly by "guess and check." Let's be systematic and set up a table. The table could have four columns, namely x, $x + 100$, $(x + 100) - 52$, $7x$. But it is clear that after the shopping Chris has \$48 more than Eric, so a three column table should do. We are guessing integer numbers of dollars. Even if the answer turns out not to be an integer, using integers should help pin things down. We want the value of "Eric" for which the entries in the

Eric	Eric plus 48	Eric times 7
0	48	0
1	49	7
2	50	14
3	51	21
4	52	28

second and third columns are equal. The third column is catching up fast, so we can just continue systematically. Or we can skip a bit. Quickly we find that Eric has \$8.

As we generate the table entries, it is useful to keep eyes open. It is quickly clear that the gap between the second and third column decreases by 6 for every dollar we add to Eric.

The gap in the first row is 48. So 48/6 additions of a dollar will bring the gap to 0. Note that "guess and check" has tuned into something dramatically different, which would work just as efficiently if instead of 48 we had, say, 9600, or indeed 46.80.

7. The greatest common factor of two numbers is 24. The least common multiple is 480. If one of the numbers is 96, what is the other number?

Solution. We can track down the other number by repeated trial. Let's give the "other number" a name. If we refer to the people in a story as this guy, that guy, the other guy, the friend of the first guy, things get confusing.

Let the "other number" be n. We know that (i) 24 is a factor of n and (ii) 480 is a multiple of n. We make a list of the possible values of n, and find out whether any of them works.

How shall we make the list? Maybe by looking at various multiples of 24, and seeing whether they divide 480. We can do this crudely, by multiplying 24 by 1, 2, 3, 4, 5, 6, and so on. This unfortunately goes on for a while, though not forever: when we multiply by 20, we reach 480, so nothing bigger will work.

Or else we can note that $480/24 = 20$, so if n is to divide 480, we can only multiply 24 by a factor of 20. Since 20 only has the factors 1, 2, 4, 5, 10, and 20, we obtain the following possibilities for n:

$$n = 24, \quad 48, \quad 96, \quad 120, \quad 240, \quad 480.$$

Now (in principle) try them all, though most can be quickly dismissed. We conclude that $n = 120$.

Another way: Students have considerable experience in finding greatest common divisors and least common multiples by factoring. Factoring is

n	$\text{GCD}(n, 96)$	$\text{LCM}(n, 96)$
24	24	96
48	48	96
96	96	96
120	24	480
240	48	480
480	96	480

grossly inefficient with large numbers, but it works well for familiar small numbers. We have

$$96 = 2^5 \times 3, \qquad 24 = 2^3 \times 3, \qquad 480 = 2^5 \times 3 \times 5.$$

Our number n can only be divisible by the primes 2, 3, and 5, since n must divide 480. We now work one prime at a time.

Start with the prime 2. Since the GCD of n and 96 is 24, the highest power of 2 that divides n is 2^3.

Look next at the prime 3. Since the GCD of n and 96 is 24, we conclude that 3 is a factor of n. But 3^2 does not divide 480, so 3^2 cannot divide n: the highest power of 3 that divides n is 3^1.

Finally, look at the prime 5. Note that 5 must divide n, since if it did not we would not have a 5 in the LCM of n and 96. And 5^2 cannot divide n: if 5^2 divided n, it would divide the LCM, but it doesn't.

We conclude that $n = 2^3 \times 3 \times 5 = 120$.

Another way: Some students "know" that the product of the LCM and GCD of two numbers a and b is equal to ab. Thus

$$24 \times 480 = 96 \times n.$$

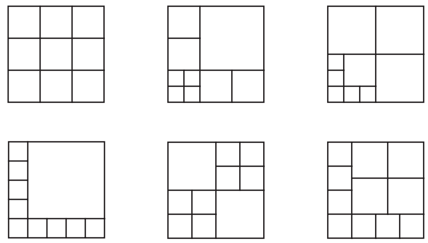
It follows that $n = 120$.

8. Show how to cut up a square into: (i) 9 squares; (ii) 10 squares; (iii) 11 squares.

Solution. There is an obvious way to split a square into 9 squares, and there are less obvious ways—please see the top 3 squares. The squares on the bottom left and bottom center are split into 10 squares. The square on the bottom right is split into 11 squares.

Comment: The square on the bottom left has been split into an L-shaped collection of 9 squares on the west and south, together with a single square to the north-east.

Suppose that the square we are splitting is 1×1, and let $k \geq 2$ be an integer. Imagine putting an "L" of width $1/k$ on the west and south, and



dividing it in the natural way into $2k-1$ squares. The picture on the bottom left illustrates the case $k=5$.

The $2k-1$ "small" squares together with the square on the north-east give us a decomposition of the original square into $2k$ squares. So we can easily cut a square into any *even* number of squares greater than 2.

What about *odd* numbers of squares? Let n be any odd number greater than 5. Then $n-3$ is an even number greater than 2, so using the "L" method we can divide the original square into $n-3$ squares. Take any of these smaller squares and split it into 4 squares. We gain 4 squares but lose 1, leaving a total of n. The decomposition into 11 squares (bottom right) was done in this way.

Thus a square can be cut into any number $n \geq 6$ of squares. Unless n is small, there are many ways of doing it. A square can also be cut into 4 squares, and "cut" into 1 square. A square cannot be cut into 2, 3, or 5 squares.

If a student finds it easy to solve the problem as stated, it is worthwhile to ask about 200 squares, 201 squares, 202 squares.

9. There are 12 whole numbers that divide 200, namely 1, 200, and 10 others. Find the product of these 12 numbers.

Solution. The number 200 doesn't have many factors, so we can list them all and multiply. Things should go OK even if the listing is not systematic. The risk of leaving out factors is small, since the number of factors has been announced.

Systematic is better. Note for example that if $ab=n$, then each of a and b, that is, a and n/a, is a factor of n. It is convenient to list the factors of 200 in such pairs a, n/a, for example as

$$1,200; \quad 2,100; \quad 4,50; \quad 8,25; \quad 5,40; \quad 10,20.$$

The product of the two numbers in a pair is 200, and there are 6 pairs, so the product of all the factors is 200^6. We now have our answer, but we may wish to multiply out. The multiplication is easy: we get 64000000000000. Actually, unless the "long" answer is really needed, it is better not to multiply. The expression 200^6 is more compact and more informative.

Comment: Let's find the product of the 15 factors of 400. Again we attempt to list the factors in pairs. Everything goes well except with the factor 20. Note that $20 \times 20 = 400$. So the factor 20 "pairs" with 20. This second 20 is not a new factor. Thus when we list the factors of 400 we get 7 pairs plus a dangling 20. The product of the 15 factors of 400 is therefore $(400^7)(20)$.

Look at the problem more generally. We want to find the product of all the factors of n. Look first at numbers n like 200 which are *not* perfect squares. Then the factors of n come in pairs. Suppose there are d factors in all. Then there are $d/2$ pairs, and each pair has product n. So the product of all the factors is $n^{d/2}$.

Look next at the case when n is a perfect square, like 400, and suppose again that n has d factors. The factors come in $(d-1)/2$ pairs, plus the unpaired factor $\sqrt{n}$, which it is more pleasant to call $n^{1/2}$. The product of all the factors is therefore $(n^{(d-1)/2})(n^{1/2})$. This turns out to be $n^{d/2}$, so actually we can use the same formula whether or not n is a perfect square.

Comment: If we know the prime factorization of n, we can calculate the number d of divisors of n without making an explicit list. We illustrate with a particular n, but the idea is general. Let n have the prime factorization

$$n = 3^4 \times 11^6 \times 31^1.$$

Any (positive) divisor k of n has the shape

$$k = 3^a \times 11^b \times 7^c,$$

where $a = 0$, 1, 2, 3, or 4, $b = 0$, 1, 2, 3, 4, 5, or 6 and $c = 0$ or 1.

To pick a divisor k of n, we first decide to what power we will take the prime 3. There are clearly 5 choices, namely 0, 1, ..., 4. For *every one* of these choices, there are 7 choices for the power to which we will take the prime 11, namely 0, 1, ..., 6. So there are already 5×7 ways of deciding the power to which 3 occurs in k *and* the power to which 11 occurs in k. Finally, decide to what power 31 will occur. For every decision about 3 and 11, there are 2 decisions about 31. So the total number of possible decisions is $5 \times 7 \times 2$.

10. A bubble tea shop carries three kinds of fruit: strawberry, mango, and pineapple. The owner of the shop allows her customers to have any combination of one or more of the above three fruits. Zero or more optional items (ice cream, coconut jelly, tapioca pearls) can be added to any bubble tea. How many different bubble teas can be ordered at this shop?

Solution. We first list and count the fruit choices. With the natural abbreviation, they are S, M, P, SM, SP, MP, and SMP. Thus there are 7 possible fruit choices.

We next count the option choices. This is just as easy, but we must remember that we can choose to have *none* of the optional items. Thus the number of option choices is 8.

There are thus 8 bubble teas whose fruit is S, 8 whose fruit is M, 8 whose fruit is P, 8 whose fruit is SM, and so on, for a total of 7×8.

Comment: The numbers above are small, so listing all fruit possibilities and then counting is the most efficient way to find the number of ways to choose the optional ingredients. We describe a fancier method of finding the number of ways of choosing 0 or more of the optional ingredients, a method that extends readily to larger problems.

Imagine the ice cream, coconut jelly, and tapioca pearls lined up in bins in front of the customer, kind of like this:

$$\mathcal{I} \qquad \mathcal{C} \qquad \mathcal{T}$$

The customer points to I and says No or Yes, or maybe in this digital age 0 or 1. Then she points to C and says 0 or 1, and finally points to T and says 0 or 1. The store owner records the customer's choices as a "3-letter word" made up of the letters 0 and/or 1. So for example 011 means no to ice cream and yes to coconut jelly and to tapioca pearls, and 000 means no to everything.

We want to count the number of 3-letter words. Obviously there are two 1-letter words. We can make a 2-letter word by adding a 0 or a 1 on the right to a 1-letter word. In this way, every 1-letter word gives birth to two 2-letter words. So there are twice as many 2-letter words as there are 1-letter words, and therefore there are 2^2 2-letter words. We make 3-letter words by adding a 0 or a 1 on the right to any of the 2-letter words. So there are twice as many 3-letter words as there are 2-letter words. The number of 3-letter words is therefore $(2^2)(2)$, that is, 2^3. The same idea shows that there are 2^4 4-letter words, 2^5 5-letter words, and so on.

Thus if there are 5 different optional items of which we can add none, some, or all, there are 2^5 different ways of deciding on the optional items.

11. In how many different ways can 11 identical muffins be distributed among Alan, Beti, and Gamal if each must receive at least one muffin?

Solution. List, or begin to list, all the ways that the muffins can be distributed. It turns out that there are 45, quite a few. Students to whom the problem appears difficult can be encouraged to look at a smaller problem, say with 7 muffins.

There are many ways to make the list systematic. A natural approach is to list all the ways in which Alan gets 1 muffin, then all the ways in which Alan gets 2 muffins, and so on up to all the ways in which Alan gets 9 muffins. Note that we are listing Beti's possible take in increasing order.

Alan	Beti	Gamal
1	1	9
1	2	8
1	3	7
1	4	6

It is clear that Beti can get anything from 1 to 9 muffins, with Gamal getting the rest. So Alan gets 1 muffin in 9 entries of our table.

Now start listing the ways in which Alan gets 2 muffins. Then Beti can get 1, 2, 3, ... 8, with Gamal getting the rest. This part of the table has 8 entries.

It is time to stop listing, and to start to *imagine* listing. There are 7 ways in which Alan gets 3 muffins, 6 in which he gets 4, and so on, up to the 1 way in which he gets 9. It follows that the total number of ways is

$$9 + 8 + 7 + 6 + \cdots + 3 + 2 + 1.$$

Add up. We get 45. It is probably worthwhile to point out the two traditional shortcuts for computing this kind of sum. Pair 9 and 1, 8 and 2, 7 and 3, 6 and 5. Pairs add up to 10, there are 4 of them, and 5 is unpaired, so our sum is 45. Or else let S be our sum. Then

$$S = 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 \qquad \text{and}$$
$$S = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9.$$

Add: we get $2S = 10 + 10 + \cdots + 10 = 90$, so $S = 45$.

Another way: Imagine the muffins laid out in a row, like this:

$$\mathcal{M} \quad \mathcal{M} \quad \mathcal{M} \quad \mathcal{M} \quad \mathcal{M} \quad \mathcal{M} \quad \mathcal{M} \quad \mathcal{M} \quad \mathcal{M} \quad \mathcal{M} \quad \mathcal{M}$$

There are 10 *gaps* in this line of 11 muffins. We will decide how many muffins Alan, Beti, and Gamal get by first *choosing* two of these gaps to put dividing lines into, maybe like this:

$$\mathcal{M} \quad \mathcal{M} \quad \mathcal{M} \quad \mathcal{M}\Big| \quad \mathcal{M}\Big| \quad \mathcal{M} \quad \mathcal{M} \quad \mathcal{M} \quad \mathcal{M} \quad \mathcal{M} \quad \mathcal{M}$$

We give everything up to the first divider to Alan, then everything up to the next divider to Beti, and the rest to Gamal. It is clear that to every way of distributing the muffins there is exactly one way of putting up dividers.

So the number of ways of distributing the 11 muffins is the same as the number of ways of *choosing* 2 gaps from the 10. This number is variously called $\binom{10}{2}$, ${}_{10}C_2$, $C(10,2)$. Surprisingly many grade 7 students know how to compute the number of games in a round robin tournament that involves 10 teams, which is simply $\binom{10}{2}$.

The idea generalizes. We want to distribute n muffins among r people so that everyone gets at least one. Line up the muffins like above. That leaves $n-1$ gaps. We want to *choose* $r-1$ of these gaps to put dividers into. The number of ways of doing this is $\binom{n-1}{r-1}$.

Comment: It is not hard to explain to students how to calculate say $\binom{10}{2}$. There are 10 people in a room, and everybody shakes hands with everybody else. How many handshakes are there? We give two approaches to the problem, the first familiar to many students, and the other less familiar.

Call the people A_1, A_2, A_3, $\dots$, A_{10}. Count first the handshakes that A_1 is involved in: clearly there are 9. Now count the ones that A_2 is involved in, and that we have *not* already counted. There are 8. Now count the ones that A_3 is involved in and that we have not already counted. There are 7. Go on in this way: the total number is $9+8+\cdots+2+1$.

We count the same thing in a more useful way. Imagine that each person keeps a journal, and puts a ✓ in her journal for every hand she shakes. Obviously A_1 ends up with 9 ✓ in her journal, and so do A_2, A_3, $\dots$, A_{10}. Thus the total number of ✓ is (10)(9). Is this the number of handshakes? No, because any handshake resulted in *two* ✓. It follows that the number of handshakes is (10)(9)/2.

Precisely the same argument shows that

$$\binom{n}{2} = 1+2+3+\cdots+(n-1) = \frac{n(n-1)}{2}.$$

An interesting variant of the muffin problem is to ask for the number of ways of distributing the muffins so that everyone gets 0 or more muffins. We can go through a listing argument that is much the same as the one given above. Or else we can use the "divider" approach. To distribute n muffins among r people so that everyone gets 0 or more, first distribute $n+r$ among them so that everyone gets 1 or more. The divider argument shows there are $\binom{n+r-1}{r}$ ways of doing this. Now take away a muffin from everyone.

12. Alan went to Oregon (which has no sales taxes) and bought two shirts, a sweater, and a pair of shoes. The shoes cost \$10.00 more than the sweater. The sweater cost \$10.00 more than each shirt. The total bill was \$129.80. How much did the shoes cost?

Solution. We could let x be the price of the shoes. Then the sweater cost $x-10$, and each shirt cost $x-20$. It follows that

$$x+(x-10)+2(x-20) = 129.80.$$

A little manipulation gives $4x = 179.80$. Since $4x$ is 20 cents short of \$180, x is 5 cents short of \$45, so $x = 44.95$.

Another way: The same idea can be carried out without any x. Suppose that after paying we change our minds and decide to buy 4 pairs of shoes instead. The sweater cost \$10 less than the shoes, so we have to throw in an additional \$10. And each shirt cost \$20 less than a pair of shoes, so we have to throw in another \$40. The cost of 4 pairs of shoes is thus \$179.80. So 2 pairs cost \$89.90, one pair costs \$44.95.

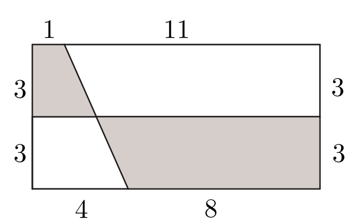
Another way: Let's guess, or pretend to guess. Since addition is a bit more natural than subtraction, we guess the price of a shirt. Let's guess 0. Unrealistic, maybe, but simple.

Check whether we are right. If shirts cost 0, the sweater cost 10, the shoes 20, for a total of 30. Not quite right, indeed 99.80 short of the truth. Make another guess, that each shirt cost 1 cent. That pushes the cost of sweater and shoes up by 1 cent, so it pushes the total bill up by 4 cents. In fact, any additional cent that a shirt costs pushes the bill up by 4 cents. We need to push the bill up by 99.80, so a shirt must cost 99.80/4.

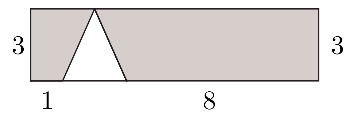
Another way: Or else we can guess, and instead of seriously thinking about the result of the guess, we can crudely classify the guess as being too small or too big. Fairly quickly we should be able to pin down the cost of a shirt, and hence of a pair of shoes, reasonably closely.

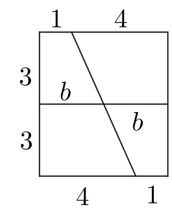
Another way: The following approach is a bit unpleasant, but maybe worth thinking about. Pick a shirt price s, and compute the total bill $B(s)$. On graph paper, plot the point $(s, B(s))$ for a few values of s. If we do it carefully, we will see that the points appear to lie on a straight line ℓ. Draw ℓ. We want to see what shirt price x corresponds to a total bill of y where $y = 129.80$. So we find graphically the x-coordinate of the point where the horizontal line $y = 129.80$ meets ℓ.

13. Find the shaded area.



Solution. Reflect the top left part in the center line, and throw away the top half of the figure.





The area shaded in the new picture is the same as the area shaded in the old one. But in the new picture the shaded region is a 12×3 rectangle with the white triangle removed. The white triangle has base 3 and height 3, so area $9/2$. The shaded area is therefore $36 - 9/2$.

Another way: The shaded area is made up of two trapezoids, one on the upper left and the other on the lower right. *If* we know how to compute the area of a trapezoid, or are willing to discover how, we should be able to find the shaded area. There is one small problem: we do not know the base of the trapezoid on the upper left. We could find it, but it turns out we don't need to.

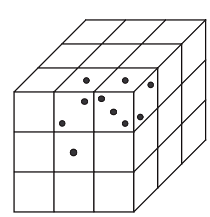
Let x be the length of the base of the upper left trapezoid. Then the two parallel sides of this trapezoid have length 1 and x, so the area of the trapezoid is $(1/2)(1+x)(3)$.

The two parallel sides of the lower right trapezoid have length $12-x$ and 8, and therefore the area of this trapezoid is $(1/2)(20-x)(3)$. Add up. The shaded area A is given by

$$A = \frac{(1+x)(3)}{2} + \frac{(20-x)(3)}{2} = \frac{63}{2}.$$

Another way: The algebraic cancellation has a natural geometric interpretation. In the picture of the problem, take the upper left trapezoid, turn it through 180°, and place it at the right of and flush against the lower right trapezoid (picture omitted). We conclude that the old shaded region has the same area as that of a trapezoid of height 3 whose two parallel sides have length 12 (upper) and 9 (lower). The area of this trapezoid can be found in various ways.

Solution. This is an exercise in visualization.



There are 3 basic types of small cubes on the visible part of the super die: (face) center cubes, corner cubes, and edge cubes.

It is easy to arrange the center dice so that they all show a 1. Since there are 6 center cubes, the center dice can be arranged so that they make a total contribution of 6.

The die face that shows 1 is not opposite the one that shows 2, so the edge dice can be arranged so that a 1 and a 2 are showing, for a total of 3. There are 12 edges, so the total contribution is 36.

Finally, since 6 is opposite to 1 and 5 is opposite to 2, the faces that show 3 and 4 are neighbours of the "1" face and the "2" face. So we can arrange the 8 corner dice so that 1, 2, and 3 are showing, for a total contribution of 48, and we can't get away more cheaply.

Add up: the smallest possible total number of dots is 90.

Comment: We can modify the problem by assembling a super die and removing the 8 corner cubes. The number of exposed die faces doesn't change, but the minimum sum does.

14. If $\mathcal{P}$ is a polygon, a *diagonal* of $\mathcal{P}$ is a line that connects two corners of the polygon but which is not a side of the polygon. Let $\mathcal{P}$ be a regular polygon with 17 sides. How many diagonals does $\mathcal{P}$ have? Note for example that a regular polygon with 5 sides (a regular pentagon) has 5 diagonals, while a regular hexagon has 9 diagonals.

Solution. Imagine that Alice, Beth, Cecil, and so on up to Quincy have been hired to stand at the 17 vertices (corners) of $\mathcal{P}$. For convenience, assume that as we travel around $\mathcal{P}$ counterclockwise, the people come in the order Alice, Beth, ..., Quincy. Alice has two immediate neighbours, namely Beth and Quincy.

There are 14 diagonals that go through Alice. (The line segments that join Alice to Beth and Quincy are *edges* (sides), they are not diagonals.) By symmetry there are 14 diagonalls that go through Beth, 14 that go through Cecil, and so on.

We must not jump to the (wrong) conclusion that there are 17×14 diagonals. To get the right answer, imagine that the Polygon corporation gives Alice 1 dollar for every diagonal that goes through her, gives Beth 1 dollar for every diagonal that goes through her, and so on. Everybody gets 14 dollars, so the cost to Polygon is 17×14. But every diagonal costs Polygon 2 dollars, 1 dollar to the person at each end. So the number of diagonals is $(17)(14)/2$.

This happens to be 119. But the answer $(17)(14)/2$ is in most ways far more satisfactory, for it captures the structure of the argument. In emphexactly the same way, we can show that if $\mathcal{P}$ is a polygon with n vertices, where $n \geq 4$, then $\mathcal{P}$ has

$$\frac{n(n-3)}{2}$$

diagonals.

Another way: Quite a few students know that for example in a round robin tournament with 10 teams, there are $9+8+\cdots+2+1$ games, and even have a reasonable idea of why. That idea can be adapted to count diagonals, but we have to be careful.

There are 14 diagonals through A. There are 14 diagonals through B, and none of these goes through A, so we are now at $14+14$. There are 14 diagonals through C, but one of them goes through A so has already been counted. We are now at $14+14+13$. Of the 14 diagonals through D, two have already been counted, so we are at $14+14+13+12$. Continue. The number of diagonals turns out to be

$$14+14+13+12+\cdots+2+1.$$

Another way: Find the number of line segments that join vertices of the polygon. So now we are including for example the line segment that joins A and B.

By an argument almost identical to the first one we gave, the number of such line segments is $(17)(16)/2$, that is, 136. The number 136 can also be arrived at as $16+15+14+\cdots+2+1$.

The count of 136 is wrong, since it includes the 17 edges (sides) of the polygon. We conclude that there are $136-17$ diagonals.

15. Candles A and B each burn at a uniform rate. But because A is thicker than B, it burns down more slowly. The candles were lit at 7:00. At 8:00, the candles were of the same height. At 11:00, candle B was finished. And at 12:00, so was A. At 7:00, A was 18 cm high. How high was B?

Solution. We will approach the problem algebraically, trying to keep things reasonably simple. We should probably measure time from 7:00, so 7:00 is time 0, 8:00 is time 1, 11:00 is time 4, and 12:00 is time 5.

Let b be the initial height of B, let r be the burning rate of B in centimetres per hour. Since B died at time $t=4$, we have $b=4r$. The burning rate of A is clearly 18/5 cm per hour.

At $t=1$, the candles were the same height. Clearly A had height $18-18/5$, and B has height $b-r$, and therefore

$$b-r=18-\frac{18}{5}=\frac{72}{5}.$$

But $b=4r$. It follows that $3r=72/5$, and now a little calculation gives $r=24/5$ and therefore $b=96/5=19.2$.

Another way: We can avoid the variables, but it takes some concentration. At 8:00 the candles were the same height, namely $18-18/5$, or $72/5$. From that time, it took 4 hours for A to die, and 3 for B. So for every 3 cm that A loses, B loses 4. Thus for every cm that A loses, B loses $4/3$. Now work backwards from 8:00. From 7:00 to 8:00, A lost $18/5$, so B lost $(18/5)(4/3)$, that is, $24/5$. It follows that at 7:00 the height of B was

$$\frac{72}{5}+\frac{24}{5}.$$

16. Find the largest positive integer n such that 2005^n is a factor of 2005!. Note that

$$2005! = 1 \times 2 \times 3 \times 4 \times \cdots \times 2003 \times 2004 \times 2005.$$

Solution. Look in turn at 2005, 2005^2, 2005^3, and so on, and test whether it divides 2005!. After a while, the answer must be no. For example, 2005^{2005} obviously does not divide 2005!, it is far too big. There are implementation difficulties with this approach. The number 2005! has 5753 decimal digits, not quite in the range of the usual calculator.

Instead, we shall imagine factoring powers of 2005, and 2005!. Factoring these numbers in easy, and imagining factoring even easier.

Note that $2005 = 5 \times 401$. The number 401 is prime. This is easy to verify. For if 401 is not prime, it must have a prime factor less than $\sqrt{401}$. But none of 2, 3, 5, 7, 11, 13, 17, or 19 divides 401.

So $2005^k = (5^k)(401^k)$ for any positive integer k. Quite high powers of 5 divide 2005!. But it will turn out that the highest power of 401 that divides 2005! is 401^5. To see that, we think about the prime factorization of 2005!.

Imagine factoring 2, 3, 4, 5, 6, ..., 2003, 2004, 2005. The prime 401 appears once in the factorizations of 401, 802, 1203, 1604, and 2005, and nowhere else, for a total of 5 times. It follows that 401^5 is the highest power of 401 that divides 2005!, and hence 2005^5 is the highest power of 2005 that divides 2005!.

Comment: We calculate in an efficient way the highest power of 5 that divides 2005!. Imagine that the numbers 1, 2, 3, 4, ..., 2005 each have to pay a tax equal to the number of 5's in their prime factorization. So a number like 24 pays no tax. Numbers like 5, 45, or 2005 pay 1 dollar. A number like 25, or 100, pays 2 dollars, while a number like 750, which is $2 \times 3 \times 5^3$ pays 3 dollars. Finally, a number like 1250 pays 4 dollars.

We collect the tax in *stages.* First collect a dollar from each multiple of 5 from 5 to 2005. There are 401 of these. The multiples of 25 still owe money. Collect a dollar from each of the 80 multiples of 25 from 25 to 2005. Then collect a dollar from each of the 16 multiples of 125 from 125 to 2005. Finally, collect a dollar from each of the 3 multiples of 625 from 625 to 2005. All the tax owed has been paid: the total is $401+80+16+3$, so the highest power of 5 that divides 2005! is 5^{500}.