UBC Workshops 2005
Solutions C

1. What can we conclude about $x + y$ from the equations

$$x^4 - y^4 = 48, \quad x^2 + y^2 = 8, \quad \text{and} \quad x - y = 2?$$

Solution. Since $x^4 - y^4 = (x^2 + y^2)(x^2 - y^2)$, we conclude from the first two equations that $x^2 - y^2 = 48/8 = 6$. Then since $x^2 - y^2 = (x + y)(x - y)$, we conclude that $x + y = 6/2 = 3$.

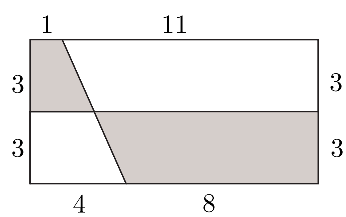
Really? Note that if $x - y = 2$ and $x + y = 3$, then $x = 5/2$ and $y = 1/2$. But then

$$x^2 + y^2 = \frac{26}{4} \neq 8.$$

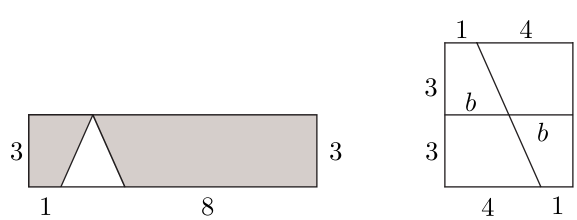
What went wrong? We assumed that a certain set of equations has a solution, say in real or complex numbers. This one doesn't—garbage in, garbage out.

Comment: This problem appeared in the 2001 Harvard–MIT Mathematics Tournament, with slightly different numbers. The published solution is wrong in the same way that the "solution" given in the first paragraph is wrong.

2. Find the shaded area.



Solution. Reflect the top left part in the center line, and throw away the top half of the figure.



The area shaded in the new picture is the same as the area shaded in the old one. But in the new picture the shaded region is a 12×3 rectangle with the white triangle removed. The white triangle has base 3 and height 3, so area $9/2$. The shaded area is therefore $36 - 9/2$.

Another way: The shaded area is made up of two trapezoids, one on the upper left and the other on the lower right. *If* we know how to compute the area of a trapezoid, or are willing to discover how, we should be able to find the shaded area. There is one small problem: we do not know the base of the trapezoid on the upper left. We could find it, but it turns out we don't need to.

Let x be the length of the base of the upper left trapezoid. Then the two parallel sides of this trapezoid have length 1 and x, so the area of the trapezoid is $(1/2)(1 + x)(3)$.

The two parallel sides of the lower right trapezoid have length $12 - x$ and 8, and therefore the area of this trapezoid is $(1/2)(20 - x)(3)$. Add up. The shaded area A is given by

$$A = \frac{(1 + x)(3)}{2} + \frac{(20 - x)(3)}{2} = \frac{63}{2}.$$

Another way: The algebraic cancellation has a natural geometric interpretation. In the picture of the problem, take the upper left trapezoid, turn it through $180°$, and place it at the right of and flush against the lower right trapezoid (picture omitted). We conclude that the old shaded region has the same area as that of a trapezoid of height 3 whose two parallel sides have length 12 (upper) and 9 (lower). The area of this trapezoid can be found in various ways.

Another way: It is in fact easy to find x. We can use analytic geometry. Take the origin at say the bottom left-hand corner, and let the positive x-axis run along the bottom of the rectangle. Then the slanted line passes through $(4, 0)$ and $(1, 6)$, so it has equation $y = (-2)(x - 4)$. Put $y = 3$. We conclude that $x = 5/2$

There are also simple geometric approaches. In the original picture, just keep the stuff to the left of the slanted line, duplicate it, turn it through $180°$ and place it as on the figure on the right. We get a 5 rectangle. By symmetry, $x = 5/2$. Now that we know x, there are several ways to calculate the shaded area.

Comment: The original rectangle was divided into two halves by the horizontal line, in order to make possible many approaches to the problem. If the horizontal line splits the rectangle into unequal parts, then it is probably easiest to compute what we have called x. Instead of being an average, x is now a weighted average. If instead of being parallel to the base, the “midline” is slanted, the algebraic approach still works in more or less the same way.

It is said that when an Eygyptian king (one of the Ptolemies) asked Euclid whether there was an easier way to approach the theorems of geometry, Euclid replied “Sire, there is no royal road to geometry.” The story is presumably false, for it is also told about Alexander the Great and his tutor, the minor mathematician Menaechmus. Dissing a king can be dangerous to your health, or at least raise problems at grant renewal time. Anyway, the story is too instructive to be true.

And in fact a royal road *is* provided by analytic geometry, as pioneered by Fermat and Descartes: coordinatize, express the geometric problem in terms of equations, and solve these equations. The process can (in principle) be automated, so a king with a large enough computing budget could prove all of Euclid's theorems.

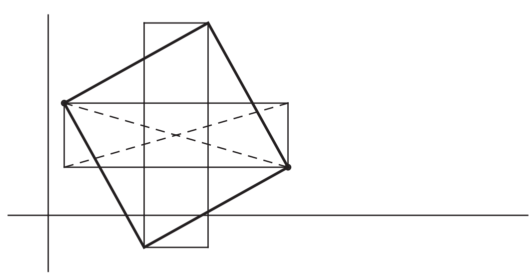
3. The vertices of square, taken counterclockwise, are $A(1,7)$, $B(s,t)$, and $C(15,3)$, and D. Find (s,t).

Solution. If we make a careful drawing on graph paper, the answer leaps out. Let's make a not so careful drawing and think about it. In going from A to B, we go right by a certain amount p, and down by a certain amount q. In symbols, $s = 1 + p$, $t = 7 - q$. In going from B to C we go up by the amount q, and right by the amount p. In symbols, $15 = s + q$, $3 = t + p$. We have the four simple linear equations

$$s = 1 + p, \quad t = 7 - q, \quad 15 = s + q, \quad 3 = t + p.$$

From the first and fourth equation, we get $s + t = 4$. From the middle two, we get $s - t = 8$. Finally, solve for s and t. We get $s = 6$, $t = -2$.

Another way: We can do the same thing without algebra. Plot the points A and C, and draw a rectangle as follows. Go straight down from A until you get to a point which is level with C (has the same y-coordinate as C), left until you get to C, up from C until you are at the same level as A, then left back to A.



Now draw a rectangle that forms a symmetrical “cross” with the first rectangle. The actual construction can be done in several ways, though it will turn out that it is enough to *think* about the construction.

The width and and height of the first rectangle can be found by counting squares, or by subtraction. Locate its center by drawing the diagonals. The second rectangle is identical to the first with directions reversed. We can draw it by counting squares from the center. Once the cross has been constructed, join points on it as shown. Symmetry shows that we obtain the desired square.

In our case, the first rectangle has base 14 and height 4, for a difference of 10. This means that the 4 pieces of the cross that stick out beyond the central square are all $(4 \times 5$. So to get to B from A, we go right 5 and down 9. The result is $(6, -2)$.

Another way: The square has central symmetry. We exploit the symmetry by finding the center. (The circle has greater central symmetry—the center is useful in almost any problem about circles.) The point halfway between A and B has coordinates $(8, 5)$. We get to A from the center by going left 7 and down 2. Thus we get to B from the center by going left 2 and down 7. It follows that $B = (6, -2)$.

Another way: The same idea can be carried out by first dragging $(8, 5)$ to the origin, where centers ought to be, by pulling it left by 8 and down by 5. Of course we have to shift the other points in the same way. Call the shifted points A', B', C', and D'.

The point A' has coordinates $(-7, 2)$, while C' has coordinates $(7, -2)$. The answer now jumps out: C' has coordinates $(-2, -7)$ (and D' has coordinates $(2, 7)$).

Push the points back to their original positions, by adding 8 to the first coordinate and 5 to the second. We find that the coordinates of B are $(6, -2)$.

Comment: We have used a simple instance of a powerful technique sometimes called "Transform, Solve, Transform Back."

Here is a more interesting geometric example. We want to find the point P on the top half of the ellipse $x^2 + 4y^2 = 1$ such that the tangent line to the ellipse at P passes through $(12, 0)$. Double y coordinates. The ellipse is transformed into the unit circle. The point P is moved to a point P' whose y coordinate is twice that of P. The tangent line to the ellipse at P is transformed into the tangent line to the circle at P', and $(12, 0)$ stays fixed.

We can now find P' by using standard properties of circles, for example the fact that the tangent line at P' is perpendicular to the radius through P'. And once we have P', we transform back by halving the y-coordinate.

The "Transform, Solve, Transform Back" idea is used throughout Mathematics, both elementary and advanced. Most "substitutions" and much of Linear Algebra can be interpreted in this way.

A	B	C
1	1	99
1	2	98
1	3	97
1	4	96

Another way: We can use more machinery. Note that B is equidistant from A and C. This yields the equation

$$(s-1)^2 + (t-7)^2 = (s-15)^2 + (t-3)^2,$$

which simplifies to the linear equation $28s - 8t = 184$. (The same equation can be obtained from the fact that the line joining B to the center is perpendicular to the line AC.)

The lines AB and BC are perpendicular, so the product of their slopes is equal to -1. This yields the equation

$$(t-7)(t-3) = -(s-1)(s-15).$$

Use the linear equation to solve for t in terms of s, and substitute for t in the quadratic above. We get a quadratic equation in s, which we then solve. Messy!

4. In how many different ways can 101 identical muffins be distributed among A, B, and C if each must receive at least one muffin? What about if there are four people?

Solution. List, or begin to list, all the ways that the muffins can be distributed. It turns out that there are 4950, quite a few. Students to whom the problem appears difficult can be encouraged to look at a smaller problem, say with 11 muffins.

There are many ways to make the list systematic. A natural approach is to list all the ways in which A gets 1 muffin, then all the ways in which A gets 2 muffins, and so on up to all the ways in which A gets 99 muffins. Note that we are listing B's possible take in increasing order. It is clear that B can get anything from 1 to 99 muffins, with C getting the rest. So A gets 1 muffin in 99 entries of our table.

Now start listing the ways in which A gets 2 muffins. Then B can get 1, 2, 3, . . . 98, with C getting the rest. This part of the table has 98 entries.

It is time to stop listing, and to start to *imagine* listing. There are 97 ways in which A gets 3 muffins, 96 in which A gets 4, and so on, up to the 1 way in which A gets 99. It follows that the total number of ways is

$$99 + 98 + 97 + 96 + \cdots + 3 + 2 + 1.$$

Add up. Daunting, perhaps, but we all know tricks to speed things up. For example, rearrange the sum as

$$(99 + 1) + (98 + 2) + (97 + 3) + \cdots + (51 + 49) + 50.$$

We get 49 100's plus 50, for a total of 4950.

Another way: The next approach is not meant to be presented, but introduces a useful technique. Let $M(n, r)$ be the number of ways of distributing n muffins among r people so that everybody gets at least one muffin. We want to compute $M(101, 3)$.

Let one of the people be Alphonse. The ways in which the muffins can be distributed among r people are of two types: (i) The ones in which poor Alphonse gets only 1 muffin and (ii) the ones in which Alphonse gets 2 or more.

How many ways are there of type (i)? The remaining $n-1$ muffins must be given to the remaining $r-1$ people. By definition there are $M(n-1, r-1)$ ways of doing this.

How many ways are there of type (ii)? Let's first toss a muffin to Alphonse, and then distribute the remaining $n - 1$ among the r people, Alphonse included, so that everybody gets at least 1 muffin. By definition there are $M(n - 1, r)$ ways of doing this. We have obtained the Muffin Recurrence Equation

$$M(n, r) = M(n - 1, r - 1) + M(n - 1, r).$$

Note that we have not obtained an *explicit* formula for $M(n, r)$, only a recipe for computing it from numbers that are presumably easier to compute. Such recurrence equations are frequently useful in combinatorics, where it can be difficult or impossible to obtain explicit formulas. Now we go back to the case $n = 101$, $r = 3$.

Note that $M(k, 2) = k - 1$ for any $k \geq 2$. (We are distributing k muffins among 2 people. The first person gets 1, or 2, or $\ldots$, or $k - 1$.)

By the Muffin Recurrence Equation

$$M(101, 3) = M(100, 2) + M(100, 3) = 99 + M(100, 3).$$

But again from the MRE, $M(100, 3) = M(99, 2) + M(99, 3)$, and therefore

$$M(101, 3) = 99 + 98 + M(99, 3).$$

Go on in this way. We conclude after a while that

$$M(101, 3) = 99 + 98 + 97 + \cdots + 3 + 2 + 1.$$

Another way: Imagine the muffins laid out in a row, like this:

$$\mathcal{M} \quad \mathcal{M} \quad \mathcal{M} \quad \mathcal{M} \quad \mathcal{M} \quad \mathcal{M} \quad \mathcal{M} \quad \mathcal{M} \quad \mathcal{M} \quad \mathcal{M} \quad \mathcal{M}$$

There are 100 *gaps* in this row of 101 muffins. We will decide how many muffins A, B, and C get by first *choosing* two of these gaps to put dividing lines into, maybe like this:

$$\mathcal{M} \quad \mathcal{M} \quad \mathcal{M} \quad \mathcal{M} \Big| \quad \mathcal{M} \Big| \quad \mathcal{M} \quad \mathcal{M} \quad \mathcal{M} \quad \mathcal{M} \quad \mathcal{M} \quad \mathcal{M}$$

We give everything up to the first divider to A, then everything up to the next divider to B, and the rest to C. It is clear that to every way of distributing the muffins there is exactly one way of putting up dividers.

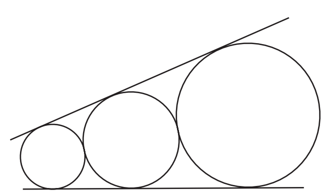
So the number of ways of distributing the 101 muffins is the same as the number of ways of *choosing* 2 gaps from the 100. This number is variously called $\binom{100}{2}$, ${}_{100}C_2$, or $C(100, 2)$.

The idea generalizes. We want to distribute n muffins among r people so that everyone gets at least one. Line up the muffins like above. That leaves $n - 1$ gaps. We want to *choose* $r - 1$ of these gaps to put dividers into. The number of ways of doing this is $\binom{n-1}{r-1}$. Now we can quickly find the number of ways of distributing 101 muffins between 4 people. The answer is $\binom{100}{3}$.

Knowing all about ${}_nC_r$ is part of the grade 12 curriculum. Most calculators have a key for computing it. One should also be able to compute by hand for reasonable n and r.

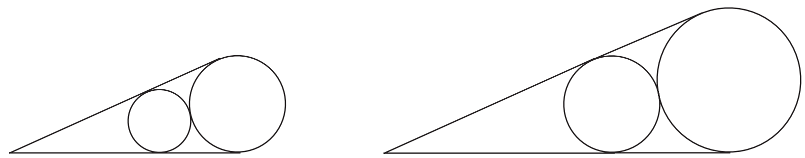
Comment: An interesting variant of the muffin problem is to ask for the number of ways of distributing the muffins so that everyone gets 0 or more muffins. We can go through a listing argument that is much the same as the one given above. Or else we can use the "divider" approach. To distribute n muffins among r people so that everyone gets 0 or more, first distribute $n+r$ among them so that everyone gets 1 or more. The divider argument shows there are $\binom{n+r-1}{r-1}$ ways of doing this. Now take away a muffin from everyone.

5. The two lines are each tangent to all three circles. The small circle has area 4 and the medium circle has area 9. Find the area of the big circle.



Solution. Most solutions use similarity. This one we give here uses similarity efficiently, with a minimum of computation.

The lines that are tangent to the circles meet in a point P. Keep only the parts of the lines that extend from P to the two points of tangency with the big circle. We get a figure that we can think of as a triple scoop ice-cream cone for the inhabitants of Flatland.



In the figure on the left, we got rid of the biggest scoop of ice-cream. In the figure on the right, we got rid of the smallest scoop. Note that the figure on the right is just a scaled up version of the figure on the left. We can get it by using a suitable zoom setting on a photocopier. The area scaling factor is 9/4, since the smaller scoop on the right has area 9 and the smaller scoop on the left has area 4. (The linear scaling factor is 3/2, but since we are computing area, the linear scaling factor is not needed.)

We conclude that the big scoop on the right has area 9(9/4).

Comment: Suppose that instead of circles between two lines we have spheres inscribed in a cone, and that the two smaller spheres have *volume* 4 and 9. By exactly the same argument as the one above, the big sphere has volume 9(9/4)

Another way: Most students will want to find the radius of the big circle in order to find the area. Part of the reason is that they are formula-oriented, and know the link between radius and area. And students are more used to working with length than with area or volume.

The two small circles have radius $2/\sqrt{\pi}$ and $3/\sqrt{\pi}$, which for convenience we call a and b. Let c be the radius of the big circle. By a similar triangles argument, one finds a relationship between a, b, and c. The most likely one to be discovered is

$$\frac{c-b}{c+b} = \frac{b-a}{b+a}$$

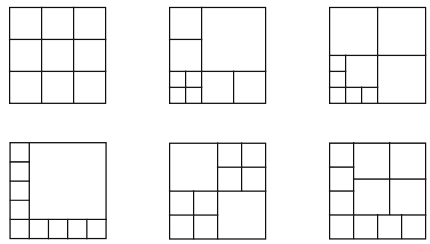
which gets transformed to $r = b^2/a$. (The natural linear scaling argument gives this more directly, in the form $r/b = b/a$).

Since a and b are known, c is known, and we can compute the area. The π “magically” disappears, although the fact that it is irrelevant to the computation may not be apparent to someone who finds a, b, and then c on a calculator.

6. Show how to cut up a square into (i) 9 squares; (ii) 10 squares; (iii) 11 squares; (iv) 2005 squares.

7. Show how to cut up a square into (i) 9 squares; (ii) 10 squares; (iii) 11 squares; (iv) 2005 squares.

Solution. There is an obvious way to split a square into 9 squares, and there are less obvious ways—please see the top 3 squares.



The squares on the bottom left and bottom center are split into 10 squares. The square at the bottom right is split into 11 squares.

Now turn to 2005. Maybe we should look at the general problem of splitting into n squares.

The square on the bottom left has been split into an L-shaped collection of 9 squares on the west and south, together with a single square to the north-east. Imagine that the square we are splitting is 1×1, and let $k \geq 2$ be an integer. Imagine putting an "L" of width $1/k$ on the west and south, and dividing it in the natural way into $2k-1$ squares. The picture on the bottom left illustrates the case $k = 5$.

The $2k-1$ "small" squares together with the square on the north-east give us a decomposition of the original square into $2k$ squares. So we have an easy way to cut a square into any *even* number of squares greater than 2.

What about *odd* numbers of squares? Let n be any odd number greater than 5. Then $n-3$ is an even number greater than 2, so using the "L" method we can divide the original square into $n-3$ squares. Take any of these smaller squares and split it into 4 squares. We gain 4 squares but lose 1, leaving a total of n. The decomposition into 11 squares (bottom right) was done in this way.

Thus a square can be cut into any number $n \geq 6$ of squares. Unless n is small, there are many ways of doing it. A square can also be cut into 4 squares, and "cut" into 1 square. A square cannot be cut into 2, 3, or 5 squares.

Another way: There is a simpler way to deal with 2005. Start with 1 square and split it into 4 squares. Then take one of the little squares and split it into 4 squares. We now have 7 squares. Take one of these squares and split it into 4 squares. We now have 10 squares. We can continue in this way, splitting the original square into 13 squares, 16, 19, and so on. What kind of numbers are we getting? All the numbers that leave a remainder of 1 on division by 3. But 2005 is such a number.

Comment: Similar ideas can be used to divide a cube into 2005 cubes. When we divide a cube into 8 cubes, we lose 1 cube and gain 8 for a net gain of 7. Since $2005 = (7)(286) + 3$, all we need to do is to find a division into n cubes where $n < 2005$ and n leaves a remainder of 3 on division by 7. After we have such an n, repeated splitting of cubes into 8 cubes will get us to 2005. Finding such an n isn't hard. For example, divide the cube into 27 cubes, and divide 5 of the little cubes into 27 cubes. We end up with 157 cubes, and $157 = (7)(22) + 3$.

8. Sketch the part of the xy-plane where $|y| + |2x - y| \le 4$.

Solution. Look first for obvious symmetries. If (a, b) satisfies the inequality, then so does $(-a, -b)$. We obtain $(-a, -b)$ by rotating (a, b) about the origin through $180°$, or by reflecting (a, b) in a point mirror at the origin, or by reflecting (a, b) in the x-axis, then reflecting the result in the y-axis.

However we visualize the symmetry, the work has been cut down by half. If, for example, we find out what the part of our region above the x-axis looks like, the rest can be filled in mechanically.

Absolute values can be difficult to work with, so it would be nice to throw away the absolute value signs. We will do exactly that, carefully.

The following slightly unnatural description of the absolute value function turns out to be useful.

$$|u| = \begin{cases} u, & \text{if } u \ge 0; \\ -u & \text{if } u < 0; \end{cases}$$

In formal treatments, the equations above are often used as the *definition* of the absolute value function. They are not needed when we deal with specific numbers, but are handy when we deal with algebraic expressions.

We will confine attention to the upper half-plane, and let reflection do the rest. So until almost the end, we assume that $y \ge 0$. Thus $|y| = y$. We have gotten rid of half the absolute value signs. Now let's look at $|2x - y|$. There are two cases to look at: (i) $2x - y \ge 0$ and (ii) $2x - y \le 0$.

(i) Case $2x - y \ge 0$: Where is this? Rewrite the inequality in the more familiar form $y \le 2x$. Draw the familiar line $y = 2x$. The region where $y \le 2x$ is the region that lies on or *below* the line. There, $|2x - y| = 2x - y$.

Thus, if $y \ge 0$ and $2x - y \ge 0$, then

$$|y| + |2x - y| \le 4 \quad \text{iff} \quad y + (2x - y) \le 4 \quad \text{iff} \quad x \le 2.$$

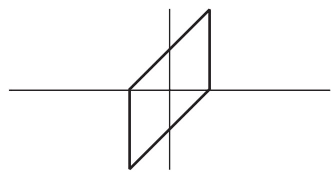
Let's interpret the result geometrically. In the part of the upper half-plane that lies below the line $y = 2x$, our inequality is equivalent to $x \le 2$. We are therefore looking at the triangle bounded by the lines $y \ge 0$, $y = 2x$, and $x = 2$. More concretely, we can describe it as the triangle with corners $(0, 0)$, $(2, 0)$, and $(2, 4)$.

(ii) Case $2x - y \le 0$: In this case, $|2x - y| = -(2x - y) = y - 2x$. So if $y \ge 0$ and $2x - y \le 0$, then

$$|y| + |2x - y| \le 4 \quad \text{iff} \quad y + (y - 2x) \le 4 \quad \text{iff} \quad y \le x + 2.$$

Let's interpret the result geometrically. In the part of the upper half-plane that lies above the line $y = 2x$, our inequality is equivalent to $y \le x + 2$. We are therefore looking at the triangle bounded by the lines $y \ge 0$, $y \ge 2x$, and $y = x + 2$. More concretely, it is the triangle with vertices $(0, 0)$, $(2, 4)$, and $(-2, 0)$.

Put things together, not forgetting to reflect in the origin, or rotate around the origin through $180°$, to get the part below the x-axis. We get



The picture shows a parallelogram with corners $(2, 0)$, $(2, 4)$, $(-2, 0)$, and $(-2, -4)$.

Another way: We can proceed more informally—but much more dangerously. We will identify the *corners* of the figure.

A look at the inequality shows that we must have $y \le 4$. Can $y = 4$? Yes, if $4 - 2x = 0$, that is, if $x = 2$. So we have identified the corner point $(2, 4)$, and, by symmetry, the corner point $(-2, -4)$.

We must also have $2x - y \le 4$. Can we have equality? Yes, if $y = 0$. We have identified the corner point $(2, 0)$. and by symmetry, $(-2, 0)$.

With a little crossing of the fingers, or by doing some fiddling with inequalities, we can decide that the parallelogram with these corners, together with its interior, is the correct region.

Another way: Let's look at the simpler problem $|u| + |v| \le 4$. When u and v are non-negative, we are looking at $u + v \le 4$, which is easy to identify: it is the region in the uv-plane on the boundary of or in the triangle with corners $(0, 0)$, $(4, 0)$, and $(0, 4)$. And since absolute value is not sensitive to sign, we get the parts in the other quadrants by reflection.

Our general strategy is to *transform* our original problem into the problem $|u| + |v| \le 4$, solve the simpler problem, and then *transform back*.

Map any point (x, y) to (u, v), where $u = 2x - y$ and $v = y$. The problem $|2x - y| + |y| \le 4$ becomes $|u| + |v| \le 4$, and the region in the uv-plane is the square with corners $(4, 0)$, $(0, 4)$, $(-4, 0)$, $(0, -4)$.

But from $u = 2x - y$, $v = y$ we obtain $x = (u + v)/2$, $y = v$ (this is the transform back phase). The square in the uv plane with corners $(4, 0)$,

$(0,4)$, and so on becomes the region with corners $(2,0)$, $(2,4)$, $(-2,0)$, and $(-2,-4)$.

Comment: The same "Transform, Solve, Transform Back" method will work just as well in similar situations, for example an inequality of the type

$$|a_1x + b_1y + c_1| + |a_2x + b_2y + c_2| \le k$$

as long as the lines $a_1x + b_1y = 0$ and $a_2x + b_2y$ are not parallel. The basic geometry is more or less the same in all cases. The region is a certain parallelogram and its interior.

9. Suppose that $x \ge 0$, $y \ge 0$, and $x^2 + y + 1 = 2xy$. Find the smallest possible value of y. No calculus please!

Solution. Rewrite the equation as $(x-y)^2 = y^2 - y - 1$. (We "completed the square." This is a standard device in problems that involve quadratics.)

The left-hand side, and therefore the right-hand side, is the square of a real number. It follows that $y^2 - y - 1 \ge 0$.

The roots of $y^2 - y - 1 = 0$ are $(1 \pm \sqrt{5})/2$, so $y^2 - y - 1$ is negative iff $(1-\sqrt{5})/2 < y < (1+\sqrt{5})/2$. So no y below $(1+\sqrt{5})/2$ can work. This does not quite finish things: just because nothing below a works does not ensure that a works.

But $y = (1+\sqrt{5})/2$ *does* work. For let $x = (1+\sqrt{5})/2$. Then $(x-y)^2$ and $y^2 - y - 1$ are both 0, so they are equal.

Another way: Since we are trying to minimize y, perhaps we should try to solve for y in terms of x. We get

$$y = \frac{x^2+1}{2x-1}.$$

Use a computer graphing program to graph the above curve. The maximum value of y can be estimated from the graph.

Another way: We solve the problem of maximizing $(x^2+1)/(2x-1)$ using standard facts about quadratics. This solution is really the same as the first solution, but sounds different.

Let b be a fixed number, and look at where the curve $y = (x^2+1)/(2x-1)$ meets the line $y = b$. This happens if $b = (x^2+1)/(2x-1)$, or equivalently if

$$x^2 - 2bx + b + 1 = 0.$$

Note that the above equation has real solutions if the discriminant $4b^2 - 4b - 4$ is non-negative, and no real solutions if the discriminant is negative.

We want to find the largest b for which the discrimant is non-negative. There, the discriminant is 0. Solve the quadratic equation for b. The larger root is given by

$$b = \frac{1+\sqrt{5}}{2}.$$

Comment: Another way of thinking about the above method is that at the maximum value of b the equation $x^2 = 2bx + b + 1 = 0$ has a "double" root. This is how Descartes, before the discovery of calculus, solved max/min and tangency problems.

10. Given that a is a number such that $\left|a - \dfrac{1}{a}\right| = 1$, what can we conclude about $\left|a + \dfrac{1}{a}\right|$?

Solution. Note that

$$\begin{aligned}
\left(a - \frac{1}{a}\right)^2 &= a^2 - 2 + \frac{1}{a^2} && \text{and} \\
\left(a + \frac{1}{a}\right)^2 &= a^2 + 2 + \frac{1}{a^2} && \text{and therefore} \\
\left(a + \frac{1}{a}\right)^2 &= \left(a - \frac{1}{a}\right)^2 + 4 = 1 + 4 = 5.
\end{aligned}$$

Finally, use the fact that use the fact that in general $|u| = \sqrt{u^2}$ to conclude that $a + 1/a| = \sqrt{5}$.

Another way: There are more awkward ways to proceed. We could solve the equation

$$\left|a - \frac{1}{a}\right| = 1, \quad \text{or equivalently} \quad a - \frac{1}{a} = \pm 1.$$

We can find the four solutions of these equations, and for every solution a compute $|a + 1/a|$. It is not quite as bad as it sounds. For a is a solution of $a - 1/a = 1$ if and only if $-a$ is a solution of $a - 1/a = -1$, so we really only need to solve one equation.

Look now at $a - 1/a = 1$. This is equivalent to $a^2 - a - 1 = 0$, whose solutions are $(1 \pm \sqrt{5})/2$. Thus

$$a + \frac{1}{a} = \frac{1 \pm \sqrt{5}}{2} + \frac{2}{1 \pm \sqrt{5}}.$$

Inserting absolute value signs gives us an answer. But it is worth trying to simplify, for example by "rationalizing the denominator." Fairly quickly we reach $\sqrt{5}$.

11. One bug is condemned to live on the circle $x^2 + y^2 - 6x - 8y = 0$, another on $x^2 + y^2 + 16x - 12y + 64 = 0$. How close can they can get to each other?

Solution. As written, the equations are insufficiently geometrical. Complete the squares in the usual way to reveal the geometry. Our equations can be rewritten as

$$(x-3)^2 + (y-4)^2 = 25 \qquad \text{and} \qquad (x+8)^2 + (x-6)^2 = 36.$$

So the first circle has center $(3, 4)$ and radius 5, while the second has center $(-8, 6)$ and radius 6.

Compute the distance between the centers, using the "distance formula." The distance turns out to be $\sqrt{125}$.

Draw the line that joins the centers. The minimum (and maximum) distance between the bugs will occur when they are on this line. The minimum distance is the distance between the centers minus the sum of the radii, that is, $\sqrt{125} - 11$.

12. Find all quadruples (a, b, c, d) of integers with $0 < a < b < c < d$ and

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} = 1.$$

Solution. This is just a matter of detective work, and some patience. Obviously a cannot be equal to 1. Almost as obviously, $a < 4$, for if $a \geq 4$, then $b \geq 5$, $c \geq 6$, and $d \geq 7$, and therefore the sum of the reciprocals of the four numbers cannot be 1. In fact, a cannot be 3. For if $a = 3$ then $b \geq 4$, $c \geq 5$, and $d \geq 6$. But

$$\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} < 1.$$

Thus $a = 2$, and our equation can be rewritten as

$$\frac{1}{b} + \frac{1}{c} + \frac{1}{d} = \frac{1}{2}.$$

If $b \geq 6$, then $1/b + 1//c + 1/d < 1/2$, so $b = 5$, 4, or 3.

First deal with $b = 5$. In that case, our equation can be rewritten as

$$\frac{1}{c} + \frac{1}{d} = \frac{3}{10}.$$

We must have $1/c \geq 3/20$, and therefore $c = 6$. But if $c = 6$, then d is not an integer.

Next deal with $b = 4$. Our equation becomes

$$\frac{1}{c} + \frac{1}{d} = \frac{1}{4}.$$

Note that we must have $c < 8$, which leaves the possibilities $c = 5$, 6, or 7. If $c = 7$, then d is not an integer. If $c = 6$, we get the solution $(2, 4, 6, 12)$. And if $c = 5$, we get the solution $(2, 4, 5, 20)$.

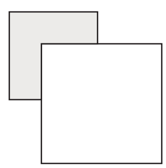
Finally, deal with $b = 3$. Our equation becomes

$$\frac{1}{c} + \frac{1}{d} = \frac{1}{6},$$

so the only possibilities for c are 7, 8, 9, 10, and 11. It turns out that 7 and 11 don't work, but the others do, and we get the solutions $(2, 3, 10, 15)$, $(2, 3, 9, 18)$, and $(2, 3, 8, 24)$.

Comment: We found 5 solutions. Perhaps they could all have been found "by inspection." But without the tedious analysis, we can't be *certain* that we have found all solutions. Systematic inspection of cases, though not much fun, is unfortunately a part of mathematics. The problem may seem "made up." But equations of this general type come up in various places, including topology and physics.

13. Glen wants to add a symmetrical L-shaped deck to his square cottage. He got a good deal on 18 metres of decorative fencing for the outside boundary of the deck (there is no fencing along the walls of the cottage). What is the largest possible area of the deck?



Solution. This is a variant of the familiar "field with a side on a river" problem of first-year calculus. We study decks that use 18 metres of fencing and have width x, and try to find the best x.

If the width of the deck is x, that takes care of $2x$ metres of fencing. The rest of the fencing $(18 - 2x))$ is shared between the two remaining fenced sides. So each has length $9 - x$.

We now compute the area $A(x)$ of the deck. An easy way to do this is to think of the deck as two x by $9 - x$ rectangles, with an x by x square removed. Thus

$$A(x) = 2x(9 - x) - x^2 = 18x - 3x^2 = 3(6x - x^2).$$

We want to find the largest possible value of $A(x)$. Calculus can be used, but there are simpler ways.

Look for example at the curve $y = 3(6x - x^2)$. We "know" that this is a downward facing parabola. The parabola meets the x-axis at $x = 0$ and

$x = 6$. "So" the line $x = 3$ is a line of symmetry, and the parabola reaches its highest point at $x = 3$. But $A(3) = 27$, and therefore the maximum possible area of the deck is 27.

Another way: The expression for $A(x)$ is a quadratic. We complete the square:

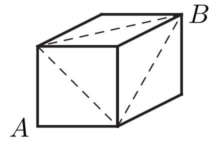
$$A(x) = 3(6x - x^2) = 3(9 - (9 - 6x + x^2)) = 3(9 - (3 - x)^2).$$

In order to make $A(x)$ as large as possible, we must make $(3-x)^2$ as small as possible. But the smallest possible value of $(3 - x)^2$ is 0, achieved at $x = 3$. That gives area 27.

Comment: The two solutions given above are actually the same. The second one is a justification of the geometric assertions made in the first.

The expression $3(6x - x^2)$ is not symmetrical enough. Completing the square brings out the hidden symmetry. We found that $3(6x - x^2) = 3(9-(3-x)^2)$. The symmetry is perhaps more obvious if we let $u = 3-x$. The area as a function of u is simply $27-3u^2$, and the symmetry is obvious.

14. The picture is of a box of the usual shape. Three face diagonals are shown; they have lengths 39, 40, and 41. Find the distance from A to B.



Solution. First we find an expression for the distance between A and B in terms of the sides of the box. Imagine heading straight up from A until we get to the dashed line that goes towards B. Let C be the point that we reach.

Let w be the height of the box, that is, the distance between A and C, and let the other two dimensions of the box be u and v.

By the Pythagorean Theorem, $(BC)^2 = u^2 + v^2$. But $\triangle ACB$ is right-angled at C. Therefore, by the Pythagorean Theorem again,

$$(AB)^2 = (BC)^2 + (AC)^2 = u^2 + v^2 + w^2.$$

This expression for the length of the "long diagonal" is pleasantly simple. Of course it is symmetrical in u, v, and w, since it is geometrically obvious that the four long diagonals are equal. The argument we used broke symmetry. A symmetrical justification would be nice.

Now we turn to our problem. Let x, y, and z be the lengths of the sides of the box. If they are listed in increasing order, we have

$$x^2 + y^2 = 39^2, \qquad x^2 + z^2 = 40^2. \qquad y^2 + z^2 = 41^2.$$

This is a system of linear equations in x^2, y^2, and z^2, so we could solve for these, and then compute $x^2+y^2+z^2$. But there is a more symmetrical way of doing things: just *add* the three equations.

By the way, the expression "add equations" does not make logical sense. An equation is an *assertion* that two things are equal. We can add *quantities,* not assertions. Unfortunately, however, "add equations" is a standard part of school language, and it is impossible to eradicate it. It is not a big problem, everybody—maybe—knows what is meant.

We arrive at

$$2(x^2+y^2+z^2) = 39^2+40^2+41^2.$$

Now just bring out the calculator. But mine isn't working. So I will note that the right-hand side of the above equation is equal to

$$(40-1)^2+40^2+(40+1)^2.$$

Imagine expanding the two ends. Note that the "middle" terms cancel, so our sum is equal to $3(40^2)+2$. Now we can do the arithmetic in our heads. It turns out that $x^2+y^2+z^2=2401$. So $AB=\sqrt{2401}=49$.

15. Alphonse ran in a cross-country race, running half of the *distance* at 3 minutes per km and half at 3 minutes 10 seconds per km. If he had run half of the *time* at 3 minutes per km, and half at 3 minutes 10 seconds per km, it would have taken him 1 second less to finish the race. How long did Alphonse actually take?

Solution. We solve a somewhat more general problem. Let half the distance be run at a minutes per km, and half at b minutes per km. Suppose that running half the *time* at a and half the time at b would have saved m minutes. We will compute the time t it actually took to run the race. In our problem, we have $a=3$, $b=3+\frac{10}{60}$, and $m=\frac{1}{60}$.

There are several reasons for working with a and b. We get increased generality. Also, letters are often less complicated to deal with than numbers. And finally and most importantly, letters may reveal structure that numerical computation hides.

It is reasonable to use "algebra." We denote certain obviously important quantities by letters, set up equations, and solve them.

Let half the distance be h. The time that Alphonse actually took was $ah+bh$, so $t=(a+b)h$.

If Alphonse had run half the time at a minutes per km, and half at b minutes per km, he would have saved m minutes. Half of what time? It would have taken him $t-m$ to finish the race, and half of that is $(t-m)/2$.

If we run for time $(t-m)/2$ at a minutes per km, we cover a distance $(t-m)/2a$. Similarly, if we run for time $(t-m)/2$ at b minutes per km, we cover a distance $(t-m)/2a$. The sum of these distances is the actual length of the race course, namely $2h$. We have obtained the equation

$$\frac{t-m}{2a}+\frac{t-m}{2b}=2h=\frac{2t}{a+b}.$$

Solve for t. We get first

$$t\left(\frac{1}{2a}+\frac{1}{2b}-\frac{2}{a+b}\right)=m\left(\frac{1}{2a}+\frac{1}{2b}\right).$$

If we are working numerically, we are essentially finished, for we can calculate the coefficient of t, and the right-hand side, and divide. But since we are working with a and b, we first simplify. Multiply through by $2ab(a+b)$. We obtain

$$t\left(b(a+b)+a(a+b)-4ab\right)=m\left(b(a+b)+a(a+b)\right),$$

and after a little more work we get

$$t=m\left(\frac{a+b}{a-b}\right)^2.$$

Finally, put $m=1/60$, $a=3$, and $b=3+\frac{10}{60}$. It turns out that $t=1369/60$, that is, 22 minutes and 49 seconds.

Comment: I don't quite know why the expression for t is so simple, but part of the simplicity is to be expected.

First imagine keeping a and b fixed. If we double the time (and therefore length) of the race, we double the gain made by running half the time at a and half at b rather than half the distance at a and half at b. And if we multiply the time by k, we multiply the gain by k. This *forces* the equation to have shape $t=mf(a,b)$.

Now imagine multiplying all of m, a, and b by k. This is equivalent to changing the unit of measurement of time, and consequently t gets multiplied by k. But t was multiplied by k if we just multiplied m by k and left a and b alone. It follows that $f(ka,kb)=f(a,b)$.

If $f(ka,kb)=k^d f(a,b)$, the function f is called *homogeneous* of degree d. So $f(a,b)$ had to be homogeneous of degree 0.

Note that we have obtained a good deal of information by quite general reasoning. The kind of analysis we have just done is called *dimensional analysis.* Dimensional analysis is important in Physics, Economics, and many other fields.

16. How many sequences a_0, a_1, a_2, a_3, a_4, a_5 of six non-negative integers are there such that (i) any number in the sequence is the sum of the previous two, and (ii) $a_5=2005$?

Solution. For simplicity let $a_0 = x$ and $a_1 = y$. Then $a_2 = x + y$, $a_3 = x + 2y$, $a_4 = 2x + 3y$, and $a_5 = 3x + 5y$. Our problem thus reduces to the number of solutions of $3x + 5y = 2005$ in non-negative integers.

Llisting all the solutions and then counting is not practical. One solution jumps out: we can take $x = 0$ and $y = 401$. Now try $x = 1, 2, 3, 4$. They don't work, but $x = 5$ does. And $x = 6, 7, 8$, and 9 don't work, but $x = 10$ does. In fact, x *must* be a multiple of 5, because 2005 and $5y$ certainly are. And any multiple of 5 should work. Not quite, we have to make sure that $3x \leq 2005$, since y cannot be negative.

So $3x$ must be a multiple of 15, and less than or equal to 2005. But 2005/15 is between 133 and 134. So the multiples of 15 between 0 and 2005 are 0×15, 1×15, 2×15, …, 133×15. There are 134 of them.

Another way: It is easier to work backwards from 2005. Let $x = a_4$. Since $a_3 + a_4 = 2005$, we have $a_3 - 2005 - x$. Similarly, $a_2 = a_4 - a_3 = 2x - 2005$, $a_1 = a_3 - a_2 = 3x - 2(2005)$, and finally $a_0 = 3(2005) - 5x$.

Note that we wrote 2(2005) instead of 4010. There is often no profit in simplifying too early: we might as well see whether our argument can be made structural rather than numerical. In fact we should probably be using the letter y instead of writing 2005. It would save bits.

The condition that the a_i are non-negative yields the inequalities

$$x \geq 0, \quad x \leq 2005, \quad x \geq 2005\left(\frac{1}{2}\right), \quad x \leq 2005\left(\frac{2}{3}\right), \quad x \geq 2005\left(\frac{3}{5}\right).$$

The first three inequalities are implied by the last two—this is obvious even without looking at the inequalities, since it is clear that if a_0 and a_1 are non-negative, so are the rest of the a_i. So we are trying to find the number of integers x that satisfy the inequalities

$$2005\left(\frac{3}{5}\right) \leq x \leq 2005\left(\frac{2}{3}\right).$$

Maybe it is time to compute. We want the number of integer solutions of the inequalities $1203 \leq x \leq 1336.66\ldots$. There are 134.

Comment: The problem is connected with a fair amount of interesting and possibly useful mathematics. The fractions that appear in the second solution are the ratios of consecutive elements of the famous *Fibonacci sequence* 1, 1, 2, 3, 5, 8, 13, 21, ….

17. The volume of the top 27 meters of an Egyptian-style pyramid is equal to the volume of the bottom 1 meter. Find the height of the pyramid.

Solution. Let the height of the pyramid be h, and let the base of the pyramid be an $s \times s$ square. Egyptian pyramids had a square base, right? Or were the bases non-square rectangles? Rhombuses, or is it rhombi?

And what is the formula for the volume of a pyramid with height h and an $s \times s$ square base? Where is my fomula sheet?

Let's see how far we can get without remembering the formula. Let the volume of the top 1 metre of our pyramid be K. Actually, by choosing our unit of volume suitably, we can let $K = 1$. So the volume of the top 1 metre of our pyramid is say 1 *pyr.* (Note that the unit of volume we use need not be the cubic metre. In the now obsolete "imperial" system, units of area not pleasantly related to square feet were used, for example the acre, and also oddball units of volume, such as the gallon.)

The top 27 metres of the pyramid are a "scaled up" version of the top 1 metre, with the linear scaling factor 27. But if you scale linear dimensions by a factor t, the volume scales by the factor t^3. Thus the volume of the top 27 metres is 27^3 pyrs.

Similarly, the top $h-1$ metres of the pyramid have volume $(h-1)^3$, the whole pyramid has volume h^3, and therefore the bottom 1 metre has volume $h^3 - (h-1)^3$. We conclude that

$$27^3 = h^3 - (h-1)^3 = 3h^2 - 3h + 1.$$

The quadratic formula gives

$$h = \frac{3 + \sqrt{9 - (4)(3)(1 - 27^3)}}{6}.$$

The height turns out to be almost exactly 81.5 metres.

Comment: The argument we used goes through, *word for word,* if the "pyramid" has an oddball base. For example, it works equally well for a cone.

Another way: We sketch a more conventional approach. Though it is less efficient than the previous solution, it is the one that students are likely to attempt. And the techniques are useful elsewhere.

Let our pyramid have height h and an $s \times s$ base. We find the size of the base of the mini-pyramid consisting of the top 27 feet.

Slice through the entire pyramid by a plane that goes through the apex and the midpoints of two opposite sides of the base. Let x be the side of the base of the mini-pyramid. By a similarity argument, we find that $x/27 = s/h$ and therefore $x = 27s/h$.

By the standard formula for the volume of a pyramid, the volume of the top 27 metres is therefore

$$\left(\frac{27}{3}\right)\left(\frac{27s}{h}\right)^2.$$

To find an expression for the volume of the bottom 1 metre, first find the volume of the *top* $h-1$ metres. By an argument which is virtually the same as the argument for the top 27 metres, we find that the volume of the top $h-1$ metres is

$$\left(\frac{h-1}{3}\right)\left(\frac{(h-1)s}{h}\right)^2.$$

and therefore the volume of the bottom 1 metre is

$$\left(\frac{h}{3}\right)s^2-\left(\frac{h-1}{3}\right)\left(\frac{(h-1)s}{h}\right)^2.$$

But the top 27 metres have the same volume as the bottom 1 metre, and therefore

$$\left(\frac{27}{3}\right)\left(\frac{27s}{h}\right)^2=\left(\frac{h}{3}\right)s^2-\left(\frac{h-1}{3}\right)\left(\frac{(h-1)s}{h}\right)^2.$$

The equation looks messy, but simplifies nicely. After a while we get the quadratic equation of the first solution.

18. Let $\mathcal{S}$ be an infinite strip of paper of width 1. What is the largest number A such that any triangle of area A can be completely covered by $\mathcal{S}$?

Solution. We give an informal argument. Sketches should be made as one reads it. Let $\mathcal{T}$ be any triangle. We want to put it on the upper half-plane so that it sticks up as little as possible, that is, so that the maximum y-coordinate of points on the triangle is as small as possible. It is reasonably clear that we should place an edge of $\mathcal{T}$ along the x-axis. And if we do, it should be a *longest* edge.

Now let $\mathcal{T}$ be a triangle that *can not* be placed on the infinite strip of width 1. Let's do the best we can, by finding the strip of minimal width on which $\mathcal{T}$ can be placed. Then $\mathcal{T}$ will be placed with a longest edge along an edge of the strip. Shrink $\mathcal{T}$ until it just fits on the strip. We get a triangle $\mathcal{M}$ that is "tightly fitting" in the sense that there are triangles of any area greater than the area of $\mathcal{M}$ that do not fit.

We want to find the minimum possible area of such a tightly fitting $\mathcal{M}$. Equivalently, we want to find the minimum possible largest side, since the height is 1.

Let $\mathcal{M}=\triangle ABC$ be a tightly fitting triangle of minimum area, with a longest edge BC lying along the edge of the strip. If $\triangle ABC$ is not equilateral, we can by moving A a little bit produce a new tightly fitting triangle (which for convenience we still call $\triangle ABC$) such that AB and AC are both less than BC. But then by moving B a little towards C, we obtain a tightly fitting triangle of smaller area. So any tightly fitting triangle of smallest area must be equilateral.

The rest is easy. An equilateral triangle of height 1 has area $1/\sqrt{3}$, so we have found A.