

UBC Workshop Solutions B

1. Three runners compete in a 100 meter race. How many possible orders of finish are there, if ties are allowed?

Solution. The runners are of course called A, B, and C. Now we must patiently list all the possible orders of finish. There are not many.

Count first the orders of finish with no ties. Any one of the 3 runners could be first. Let's count how many orders of finish there are with say A first: clearly there are 2. And there are 2 with B first, 2 with C first, for a total of 6.

Now count the number of orders of finish with exactly 2 runners tied. They could be (i) tied for first or (ii) tied for last.

To count the orders of finish with two people tied for first, think of the last runner. She can be chosen in 3 ways. Once that is done, we are finished. Similarly, there are 3 orders of finish with exactly 2 runners tied for last, for a total of 6.

And of course there is the possibility of a triple tie. Thus overall there are $6 + 6 + 1$ possible orders of finish.

Comment: If someone finds three runners too simple, perhaps she can look at four runners (the answer is 75). The counting can be done as follows.

The number of orders of finish with no ties is 24 ($4!$). For the number of orders of finish with a single tie, the people to be tied together can be chosen in $\binom{4}{2}$, that is, 6 ways. Then we can think of them as three people, and they can finish in $3!$ orders, giving a total of 36 ways. There are 4 ways of choosing three people to be tied together, and for each there are 2 orders of finish, for a total of 8 ways. And there is the 1 quadruple tie. The counting gets tricky when we deal with the number of ways that we can have a "tie for first and tie for last." Here we choose the pair of people who will tie for first (6 ways). Once this is done we are finished, we *must not* double.

2. An arena has 21000 seats. It is divided into four sections. Section A has twice as many seats as Section B. Section C has twice as many seats as Section D. Section B has 1000 more seats than Section D. How many seats are in each section?

Solution. For people who are comfortable with basic algebra, it is natural to let x be something or other, maybe the number of seats in D—that way

we can avoid subtraction and fractions for a while.

So C has $2x$ seats, B has $x + 1000$, A has $2x + 2000$. Add up. We get $6x + 3000 = 21000$, so $x = 3000$ and we are finished.

Or else we could start with four letters a , b , c and d , write down the obvious equations and simplify. But at this stage students are not comfortable with several variables. This discomfort has long historical roots. Diophantus and early Islamic mathematicians such as al-Khwārizmī dealt with some degree of comfort with a single “unknown.” But many hundreds of years elapsed before mathematicians handled several variables with confidence.

Another way: We could also probably guess and refine. This works fast since the answer is too simple. Maybe guess that D has 0 seats? Then C also has 0, B has 1000, A has 2000, for a total of 3000, not enough. Maybe now guess 1000 for D. That gives arena size 9000. In fact adding 1 person to our guess for D adds 2 to C, 1 to B, 2 to A, so adds 6 people. We want to get from 3000 to 21000, need to add 6 people 3000 times, giving 3000 people for section D. (A very good guess for the size of D is -500 .)

Another way: Or else first deal with the 1000 “extra” seats that B has, and therefore the 2000 extra that A has. Remove these, we have 18000. And now A and B clearly balance C and D, so C and D have 9000, and therefore D has 3000.

There are other ways of reasoning the problem through. A picture can be useful.

3. A sports league has two conferences, East and West. Each conference has 10 teams. Every year, each team plays every team in its conference twice and plays every team in the other conference once. What is the total number of games played in the league during the year?

Solution. First look at the total number of games between teams of the Western Conference. Since each team plays every team in its conference twice, we can think of these two games as being “home” and “away.”

Call the teams of the Western Conference $W_1, W_2, W_3, \dots, W_{10}$. Looks nice on a team sweater. Let’s count the number of *home* games between Western Conference teams. Since any game is a home game for one of the teams, we will have counted all games between Western Conference teams. Each team, like W_1 , plays 9 home games against Western Conference Teams. But there are 10 teams, so the number of games is 90.

By symmetry there are 90 games between Eastern Conference Teams.

Now we need to count the number of inter-conference games. Team W_1 plays 10 such games, as does W_2 , and so on down to W_{10} , a total of 100 games. But this is *all* the inter-conference games.

Thus the total number of games is $90 + 90 + 100$.

Suppose more generally that the Western Conference has m teams and the Eastern Conference has n teams. The same argument shows that the total number of games is

$$m(m-1) + n(n-1) + mn.$$

More generally suppose that any two teams in the Western Conference play a games with each other, any two teams in the Eastern Conference play b games with each other, and any two teams in different conferences play c games. Then the total number of games is

$$\frac{am(m-1)}{2} + \frac{bn(n-1)}{2} + cmn.$$

Another way: It is informative to view things more geometrically. To count the number of games between Western Conference teams, draw a 10×10 array with rows and columns labelled with the team names. The two games between every pair of teams are represented by the 90 entries in the array (100 minus the 10 “diagonal” entries). If there were only one game for each pair of teams, the games could be represented for example by the 45 entries in the triangle above the main diagonal.

Representing the inter-conference games is even simpler: label the rows of a 10×10 array with the Western Conference team names, and the columns by the Eastern Conference team names.

Another way: Here is an argument that is at once simpler and harder. Look at a particular team, say W_1 . This team plays 2 games against each of the 9 Western Conference teams (18 games) and 1 each against the 10 Eastern Conference teams, for a total of 28 games. Each team in the league does the same thing, for a total of $(28)(20)$. Well, not really. When we multiplied 28 and 20, we were counting twice each game that team W_1 played, once from W_1 's point of view and once from the other team's point of view. So the total number of games is $(28)(20)/2$. Note that this approach uses the symmetry between the two conferences.

4. A, B, C, and D are running a marathon along a straight road. As usual, A is in front, B is next, C is behind B, and D is behind C. At this instant, A is 1 mile ahead of C, B is 4 times as far from A as she is from C, and D is also 4 times as far from A as she is from C. What is the distance, in miles, between B and D?

Solution. If we look at the space between A and C (one mile) then maybe it is clear, if we draw a little picture, that C is one-fifth of a mile behind B, and B is four-fifth of a mile behind A.

Now look at the relationship between A, C, and D. So from A to D is four ‘parts’ while A to C is 3 parts. But A to C is one mile, so A to D is 1

mile plus $1/3$ of a mile. Because we are dividing a mile into 5 parts, and also into 3 parts, it may be a good idea to break up the mile into 15 parts. The rest is easy. Draw a line to represent the 1 mile distance between A and C, and break it up into 15 equal parts. Then the rest can be figured out: B to D is 8 parts, eight-fifteenths of a mile.

We can also use “fractions.” Fine of course, but a bit further from basic intuition.

5. A poster is 40 centimeters wide. There are two pictures on the poster. Each picture is 25 cm wide and 20 cm high. Together the pictures take up one-third of the area of the poster. How many centimeters are in the height of the poster?

Solution. The area taken up by each picture is 25×20 , namely 500 (square centimetres), so the two pictures together have area 1000. This is one-third of the area of the poster, so the poster has area 3000. It has width 40, so it has height $3000/40$, that is, 75. Students can be guided to do mental arithmetic: $3000/40 = 300/4 = 150/2 = 75$.

6. How many positive integers are factors of 720? Here are a few of them: 1, 5, 8, 360, 720.

Solution. It turns out that there are 30 of them, so if we are going to make a list it should be an efficient list. Maybe we should first factor 720:

$$720 = 2^4 \times 3^2 \times 5.$$

First look at the factors of 720 with no 2's in them, that is, factors of $3^2 \times 5$. These perhaps can be listed explicitly: 1, 3, 5, 9, 15, 45, six of them

Then look at the factors of 720 that have exactly one 2 in them. These are all numbers which are twice numbers in our first list, explicitly 2, 6, 10, 18, 30, 90, but we don't need to list them to see there are six.

Then look at the factors of 720 that have exactly two 2's in them. These are all numbers which are twice a number in the preceding list, or four times a number in the first list, in all six numbers.

Then look at the factors of 720 that have three 2's. There are six. And there are six that have four 2's. So the total is $6 + \cdots + 6$ (five sixes, 30).

Another idea, a bit unpleasant but not too bad, is to note that since $27 \times 27 = 729$, whenever we express 720 as $a \times b$, the smaller of a or b must be less than 27, and the bigger must be bigger than 27. So the factors of 720 come in pairs (a, b) where a is less than 27 and $b = 720/a$. Now start listing all the factors less than 27: 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 16, 18, 20, 24, fifteen in all. Then there are their fifteen mates, for a total of 30.

Or else we start from $720 = 2^4 \times 3^2 \times 5$, and make up a divisor of 720 by first deciding how many 2's we want (we can have 0 to 4 of them, 5 choices). For every choice of how many 2's, we have 3 choices for how many 3's (0, 1, or 2), and 2 choices for how many 5's (0 or 1). The total number of choices is thus 5×2 .

It may be interesting to some students how one can find the number of positive integer divisors of n where

$$n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$$

where the p_i are distinct primes. The same argument as the one above shows that there are $(e_1 + 1)(e_2 + 1) \cdots (e_k + 1)$ divisors.

7. A gambler is allowed to toss a fair coin six times. She wins if during the tossing she gets three or more heads in a row or three or more tails in a row. What is the probability that the gambler wins?

Solution. To solve this problem we need to know how many different “outcomes” there are, that is, how many strings of length 6 made up of the letters H and/or T. There are 2^6 such strings. A good way of showing this is with a tree diagram.

Now we ask how many strings there are that have 3 or more heads in a row or three or more tails in a row.

It is in principle not hard to count, but it would be nice to do the counting reasonably efficiently.

Half of our strings start with H and half start with T. There is complete symmetry, so we will count how many ways we can start with H and get 3 or more H or T in a row. And we will double the result to take care of the strings that start with T.

We start with H. Maybe we get lucky and the first three tosses are H. The number of strings of this type is 2^3 , for after our three heads any pattern of H and/or T is allowed. This gives 8 ways.

Maybe we got two H in a row and then (disappointingly) a T. We can still win if (i) we now get three H in a row (1 way) or we get two more T, and then anything (2 ways), for a total of 3 ways.

Maybe we got an H and then a T. If the next letter is an H, we are at HTH, and now a couple of HH would be nice (2 ways, for last letter does not matter) or three T (1 way). If the next letter is a T, we are at HTT, and another T will do the job (4 ways, for we don't care about the last two letters), or three H (1 way). The total here is thus 8.

So we have 19 winning patterns that begin with H, another 19 that begin with T, for a total of 42. So the probability our gambler wins is $38/64$.

Note that the probability is 59.375, likely higher than one might have guessed. Winning and losing streaks are highly likely on purely probabilistic grounds; sports fans assign to them much more significance than they probably deserve.

8. Tom has a total of \$3.30 in nickels and dimes. If his nickels were dimes and his dimes were nickels, he would have \$4.80. How many nickels and how many dimes does Tom have?

Solution. We can solve the problem with least intellectual effort by using “algebra.” Let x be the number of nickels and y the number of dimes. Then the amount of money (in cents) is $5x + 10y$. If nickels were dimes and dimes were nickels then the amount of money would be $10x + 5y$. We conclude that

$$\begin{aligned} 5x + 10y &= 330 & \text{and} \\ 10x + 5y &= 480. \end{aligned}$$

There are many ways of solving this system of equations. The most primitive is to note from the first equation that $5x = 330 - 10y$ and therefore $10x = 660 - 20y$. Substituting in the second equation we obtain $660 - 20y + 5y = 480$, and therefore $15y = 180$, so $y = 150/15 = 12$. But then $5x = 330 - 120 = 210$, so $x = 42$.

Or else we can add the two equations and obtain $15(x + y) = 810$. It follows that $5x + 5y = 270$ and therefore from $5x + 10y = 330$ we conclude that $5y = 330 - 270 = 60$, so $y = 12$.

Another way: We can reason our way to an answer. It is clear that there are more nickels than dimes (else when identities are switched the amount of money goes down). So imagine a stack of dimes and a taller stack of nickels. For simplicity of visualization let dimes and nickels have the same thickness—they don’t.

When nickels are changed to dimes and dimes to nickels, the increase in value is caused by the “extra” nickels, and is 5 cents for every extra nickel. But the value climbed by $480 - 330$ cents, that is, 150 cents. So the number of extra nickels was $150/5$, namely 30.

The rest of the \$3.30, namely $330 - 150$ cents, is evenly split between nickels and dimes. How many of each? Weld each nickel to a dime to make a 15 cent piece. We would have $180/15$ of them, that is, 12. So Tom has 12 dimes and 42 nickels.

Another way: Or else imagine that Tom has the \$3.30 in his left pocket, and the “switched” version in his right pocket. So he has a total of $3.30 + 4.80$, that is, 810 cents, and an equal number of nickels and dimes. Weld a nickel and a dime together to make a 15 cent coin. We have $810/15$ of them, that is, 54.

So Tom started off with 54 coins. If they had been all nickels he would have only had \$2.70. But he had \$3.30, 60 cents extra. Trading in two nickels for a dime gains 5 cents, so 12 trades are required to get to \$3.30. It follows that there were 12 dimes and therefore 42 nickels.

Another way: Or else let's "guess" what the number of nickels is, and then make an improved guess, and then perhaps another, until we reach the answer. Ideally we would make only a small number of guesses, and they should not really be guesses.

Let us guess for example that the coins are all nickels. This is wrong, because if they were, then when nickels are changed to dimes and dimes to nickels, we would have \$6.60, not \$4.80. So imagine trading in two nickels for a dime, maybe repeatedly. Any such trade leaves us at \$3.30, but when identities of coins are switched, we lose 15 cents, 10 cents for the nickels that did not become dimes, and 5 cents for the dime that became a nickel. We want to go down from \$6.60 to \$4.80, so by 180 cents. The number of trades is therefore $180/15$, that is, 12, and Tom has 12 dimes.

9. Let $x * y = \frac{x}{x+y}$. If $x * y = 9$, what is $y * x$?

Solution. We start with the equation $x/(x+y) = 9$. We can manipulate our way to success. For $x = 9x + 9y$. Thus $9y = -8x$ and therefore $y = (-8/9)x$.

Now we can find $y * x$, that is, $y/(x+y)$. Note that $x+y = x + (-8/9)x = x/9$, and therefore

$$\frac{y}{x+y} = \frac{(-8/9)x}{x/9} = -8.$$

Another way: We can do the same thing in a slightly easier way. Note that $x * y$ and $y * x$ are both determined by the *ratio* of x and y . So we can choose x or y or $x+y$ almost freely: of course we must not have $x+y = 0$. For the sake of symmetry, let's pick $x+y$, say it is 1. Then from $x * y = 9$ we conclude that $x = 9$. But then $y = -8$. But then $y * x = -8$.

Another way: Even this is too complicated. Note that

$$(x * y) + (y * x) = \frac{x}{x+y} + \frac{y}{x+y} = 1$$

so $x * y = 1 - 9 = -8$.

Another way: We could work harder, maybe like this. Since $x/(x+y) = 9$, we have $(x+y)/x = 1/9$, so $1 + y/x = 1/9$ and therefore $y/x = -8/9$. Now work backwards. We have $x/y = -9/8$ and therefore $1 + x/y = -1/8$ and therefore $y/(x+y) = -8$.

10. The plane was full when it left Vancouver. In Seattle, half the people got off and 28 got on. In Portland, half the people got off, 40 got on, and the plane was full again. How many people were on the plane when it left Vancouver?

Solution. It is not unreasonable to find our way by experimenting. Hard to know what guess we would start with. Say 40. The 48 people leave Seattle, and therefore 64 leave Portland. Is 40 too big or too small? Maybe it is not obvious.

Try for what next? Maybe 41? So $41/2 + 28$ leave Seattle. Possible but perhaps a bit gruesome. And 42 gives us the same problem of half a person when we leave Portland. And of course 43 is no good. Try 44.

So 50 leave Seattle, 65 leave Portland. Still not good, but 44 is closer to 65 than 40 was to 64. Maybe try for 48 out of Vancouver. Then get 66 out of Portland. Interesting pattern! We add 4 to the number leaving Vancouver, and the number out of Portland rises only by 1, so the “gap” went down by 3.

With this observation we can finish things. At 40 the gap between Vancouver and Portland was 24. Going up by 4 in Vancouver closes the gap by 3. We need eight 3’s to get rid of the gap of 24. So in Vancouver we should go up by 8 4’s, that is by 32, up to 72. Check. It works.

A good way to handle things is with proto-algebra. Maybe represent the number of people leaving Vancouver by a line. Then out of Seattle come half the line plus 28. Out of Portland come half of half the line plus 14 plus 40, that is, a quarter of the line plus 54. And this is the whole line. Then we can do arithmetic, but maybe it is better, since one-quarter of the line comes into the game, to think of the people leaving Vancouver (the original line segment) as being made up of 4 equal line segments. After a while we get that 3 of the chunks add up to 54 so one chunk is 18 and 4 chunks give 72.

Of course we could go through the x business. So there are x when we leave Vancouver, then $x/2 + 28$ leave Seattle, and $1/2(x/2 + 28) + 40$ leave Portland. But this is x . So

$$1/2(x/2 + 28) + 40 = x.$$

How to solve this is maybe not obvious. One way is to multiply through. But another, for the fraction-challenged, is to “unravel” it, $1/2(x/2 + 28) = x - 40$, so $x/2 + 28 = 2x - 80$, so $x/2 = 2x - 108$, so $x = 4x - 216$, so $3x = 216$.

11. If n is a positive integer, then $n!$ (read this as “ n factorial,” or “factorial n ”) is the product of all the numbers from n down to 1. For example,

$$4! = 4 \times 3 \times 2 \times 1 = 24, \quad 5! = 5 \times 4 \times 3 \times 2 \times 1 = 120.$$

Find the highest power of 2 that divides $32!$. For example, the highest power of 2 that divides $5!$ is 2^3 .

Solution. We have

$$32! = 1 \times 2 \times 3 \times 4 \times 5 \times \cdots \times 30 \times 31 \times 32.$$

We can see how many 2's there are "in" each of 1, 2, 3, 4, 5, ..., 32, and add up.

There are no 2's "in" 1, 3, 5, 7, ..., 31. The number 2 has one two in it (1), the number 4 has two 2's in it (2), the number 6 has one (1), the number 8 has three (3), the number 10 has one (1), the number 12 has two (2), and so on. Add up. It is not too bad.

We can save quite a few steps by noting that the numbers 2, 6, 10, 14, 18, 22, 26, 30 each have one 2, for a total of 8. And 4, 12, 20, 28 have two, for a total of 8. And 8, 24 have three, for a total of 6. And 16 has 4, and 32 has 5. Add; we end up with 31. So the highest power of 2 that divides $32!$ is 2^{31} .

There is a much nicer way of looking at things. Imagine that each of the numbers from 1 to 32 has to pay a one-dollar tax for each 2 in it. So for example the number 8 pays a tax of \$3.

Collect one dollar from each of the $32/2$ even numbers from 1 to 32. Some numbers, like 4, 8, and so on still owe tax. Collect a dollar from each of the $32/4$ multiples of 4 from 1 to 32. The numbers 8, 16, and so on still owe money. Collect a dollar from each of the $32/8$ multiples of 8, a dollar from each of the $32/16$ multiples of 16, and finally a dollar from each of the $32/32$ multiples of 32. Now everyone has paid the proper tax, which adds up to

$$32/2 + 32/4 + 32/8 + 32/16 + 32/32$$

that is 31.

Let p be a prime. A mild modification of the idea above enables us to calculate the highest power of p that divides $n!$.

- 12.** A paper drinking cup is cone-shaped. When there is water in the cup to a depth of 4 inches, the cup contains 16 cubic inches of water. How many cubic inches of water are in the cup when the water is 3 inches deep?

Solution. It is tempting to first reach for a formula, and then for a calculator. If a cone has height h and 'base' radius x , then its volume V is given by

$$V = \frac{\pi x^2 h}{3}.$$

It would be easy to compute the volume of water when the depth is 3 inches if we knew the base radius r . We don't know r , but we can compute the base radius R when the depth of water is 4 inches.

When the water cone has height 4, our cup contains 16 cubic inches. It follows that $16 = 4\pi R^2/3$, and therefore

$$R = \sqrt{\frac{(16)(3)}{4\pi}}.$$

Comment: At this point many would reflexively reach out for the calculator and compute R to some number of decimal places. This is a *very bad* habit. Our expression for R has *structure*. Pushing it through a calculator turns a structured object into a jumble of digits.

The two water cones are *similar*. It follows that

$$\frac{r}{R} = \frac{3}{4}$$

and therefore $r = (3/4)\sqrt{(16)(3)/(4\pi)}$.

Now we can compute the volume of water, namely $3\pi r^2/3$. If we use the value of r found above, we get quickly that the volume is $(3)(3/4)^2(16)/4$ (the π 's cancel). Now, with calculator or without, we find that the volume is 6.75 cubic inches.

Another way: There is a *much* better way of looking at the problem—so much better that the solution we have just given should be called the wrong solution.

The ‘small’ water cone is just a scaled down version of the big water cone. The *linear* scaling factor is $3/4$, that is, all lengths get multiplied by $3/4$. If we multiply the dimensions of an object by the linear scaling factor t , then areas scale by the factor t^2 , and volumes scale by t^3 . So the volume of our small water cone is $(3/4)^3(16)$.

- 13.** Ten consecutive odd integers add up to 800. What is the smallest of these integers? An example of 10 consecutive odd integers is 7, 9, 11, 13, 15, 17, 19, 21, 23, 25—but they don’t add up to 800.

Solution. Here is an ugly solution. Let the smallest number be x . Let’s not worry too much about whether it is odd or not. Then the others are $x + 2$, $x + 4$, $\dots$, $x + 18$. Add up, we get $10x + 90$, which is 800, giving $x = 71$. Since 71 is an odd integer, 71, 71 + 2, 71 + 4, $\dots$, 71 + 18 are consecutive odd integers, and they have the right sum, so the answer is 71.

Comment: The last sentence is not quite superfluous. Let us change the problem slightly to “Ten consecutive odd integers add up to 790. What is the smallest of these integers?” A calculation almost identical to the one above gives $x = 70$, which is clearly wrong, 70 is not odd. In general when we write “Let $x = \dots$ ” we are making the implicit assumption that an

object with the desired properties exists. If this is not true, conclusions that we draw about x will be wrong.

Another way: Here is something a little more attractive. Look at the two middle numbers, and let x be the number halfway between them. So the numbers are $x - 9, x - 7, x - 3, x - 1, x + 1, x + 3, x + 5, x + 7$. Add up, grouping in the obvious way. We get $10x = 800, x = 80$, so our numbers are all odd and the smallest is $80 - 9$.

Another way: Or else “guess” that the numbers are 1, 3, 5, 7, 9, 11, 13, 15, 17, 19. Add up, we get 100, bad guess, the sum is off by 700. So add 70 to each number, the sum is right!

Another way: A cute (maybe too cute) idea is to “guess” that the numbers are $-9, -7, -5, -3, -1, 1, 3, 5, 7, 9$. Spectacularly bad guess, but the sum is easy to find, it is 0. If we add 80 to each number we get the right sum.

Another way: Or else we can start from the sequence 80, 80, $\dots$, 80 (ten 80's), which has the right sum but does not consist of consecutive odd integers. We modify this sequence so as to keep the sum unchanged while satisfying the “consecutive odd” condition. Change the two 80's closest to the middle to 79 and 81. So now we are at

$$80, \quad 80, \quad 80, \quad 80, \quad 79, \quad 81, \quad 80, \quad 80, \quad 80, \quad 80.$$

Change the two 80's closest to the middle to 77 and 83, then the two remaining 80's closest to the middle to 75 and 85, and so on until we get to the sequence 71, 73, 75, 77, 79, 81, 83, 85, 87, 89.

Comment: It is worthwhile to think about generalizing. Let's stick to consecutive odd numbers, but look for n consecutive odd numbers whose sum is S .

If there are such numbers, then n must be a factor of S . For if n is even we can without loss of generality take the consecutive odd numbers to be $a \pm 1, a \pm 3$, and so on. Their sum is an , a multiple of n . And if n is odd we can take the numbers to be $a, a \pm 2$, and so on, again with sum an .

Now assume that S/n is an integer. Can we automatically find n odd consecutive integers with sum equal to S ? Not necessarily: the sum of an odd number of odd numbers is odd, so if n is odd we must have S odd. And if n is even, and we let the numbers be $a \pm 1, a \pm 3$, and so on, then $a = S/n$, and therefore we are forced to have S/n even.

Apart from these restrictions, there is no problem. If n is odd, and S is odd, and S/n is an integer (necessarily odd), we can use the numbers $a, a \pm 2, a \pm 4$, and so on for a total of n numbers. And if n is even and S/n is even, let $a = S/n$ and use $a \pm 1, a \pm 3$, and so on for a total of n numbers.

We can extend these observations to sums of consecutive members of an arithmetic progression of integers with common difference d (we have just dealt with $d = 2$).

By the way, the following problem has been discussed on the Secondary math teachers' listserv. Is it true that every positive integer which is not a power of 2 can be expressed as the sum of two or more consecutive positive integers? (Yes.) In how many ways?

14. How many ways are there to write down three numbers a , b , and c chosen from the numbers 1, 2, 3, 4, $\dots$, 8, 9 so that $a < b < c$ and $a + b + c$ is a multiple of 3? (One such triple of numbers is 4, 6, 8.)

Solution. We can make a list in some ad hoc way. There are some problems with this. The process is uninteresting; it is error-prone; and if we want to deal with a number significantly greater than 9, the listing becomes very tedious.

There are various ways to *imagine* listing rather than actually listing. Here is one way.

We can think of the process of writing down the numbers as taking place in two steps: (i) We *choose* three numbers whose sum is a multiple of 3 and then (ii) We call the smallest one a , the second smallest b , and the largest c .

Once we have done step (i), step (ii) can only be done in one way. So let's just look at step (i).

It is very useful to realize that the numbers 3, 6, and 9 are roughly speaking interchangeable, since they are all divisible by 3. These numbers will be called numbers of type 0. Similarly, 1, 4, and 7 have the same remainder on division by 3. Call them numbers of type 1. And finally call the numbers 2, 5, and 8 numbers of type 2.

Three numbers x , y , and z have sum divisible by 3 if either they are all of the same type, or they are all of different types.

Let's first count how many ways we can choose 3 numbers all of the same type. This is easy. The type is either 0, 1, or 2. And since there are only 3 numbers of each type, we have to pick all 3. So there are 3 ways to choose 3 numbers all of the same type.

Now let's see how many ways there are to choose 3 numbers of different types. First let's choose the number of type 0; this can be done in 3 ways. The number will be 3, 6, or 9.

We can draw this as a tree. From a "root" (placed on top!), draw 3 lines, and label the ends, or the branches, 3, 6, and 9. Now look at the branch labelled 3. We can pick the number of type 1 in 3 ways. Do the same with the other two branches. Continue. We end up with $3 \times 3 \times 3$ possibilities.

One can think of it as a mother (the root) having 3 children, and then each child has 3 children, and each of these has 3 children. The root has

3^3 great grandchildren. So there are 27 ways of picking three numbers of different types.

It follows that there are $3 + 27$ triples with the desired properties.

15. The interior of cooking pot A is a cylinder with base diameter 15 cm and height 10 cm. The interior of cooking pot B is a cylinder with base diameter 30 cm and height 40 cm. Pot A is filled with water and the contents are poured into pot B. After this has been done a total of six times, how many cm deep is the water in pot B?

Solution. We can trot out the machinery of volumes. The volume of the small pot is $\pi(15/2)^2(10)$. If water is h centimetres deep in the big pot, then the volume of water is $\pi(30/2)^2h$. So to find how much one small potful contributes to height in the big pot, we set

$$\pi(15/2)^2(10) = \pi(30/2)^2h$$

and after some cancellation (or a calculator computation) find that $h = 10/4 = 2.5$. So 6 potfuls give depth of 15 centimetres.

Another way: The big cooking pot has diameter twice the diameter of the little one, so the area of the bottom of the big pot is 4 times the area of the bottom of the little one. That means that 10 cm of water in the little pot only give us $10/4$ in the big pot. Repeat 6 times.

Note that everything would be intuitively obvious if pots had square bases—four potfuls from A give us 10 cm in pot B.

16. Three swimmers had a race across a small lake. Each swam at constant speed. When A finished, she was 20 metres ahead of B, and 40 metres ahead of C. When B finished, she was 20.5 metres ahead of C. Over how many metres was the race?

Solution. When A finished, B had 20 metres to go, and had a 20 metre lead on C. While swimming 20 metres she built up her lead by 0.5 metres.

But B had built a lead of 20 metres over C. Since she increases the lead by 0.5 metres for every 20 metres she swims, or by 1 metre for every 40, she needs 40×20 to build a lead of 20 metres. That gets B to 20 metres from the end, so the length of the race is 820 metres.

17. Al and Bob are having a two lap race in a 30 metre pool. Al swims the first lap freestyle at 2 metres per second. For the second lap he swims the backstroke at 1 m/s. Bob swims the butterfly at 1.5 m/s for the entire race. At what time(s) after the start will Al and Bob be side by side? (Al and Bob are very small—in fact they are points.)

Solution. Bob swims 60 metres at 1.5 metres per second, so he takes $60/1.5$ seconds, that is, 40 seconds. Al takes $30/2$ seconds for the first lap and $30/1$ for the second, for a total of 45. Al loses.

But it is clear that Al finishes the first lap well ahead of Bob. So Bob and Al must meet once while Al is already on his second lap and Bob is still on his first. And they must meet again before the end of the race, since Bob will win.

“Algebra” gives us a fairly mechanical way of finding when they meet. But the algebra must be guided by a geometric view of the race.

Let’s suppose Al and Bob first meet after t seconds. We have seen that Al is already on his second lap, so $t > 15$. Al has travelled a length of the pool, plus $(1)(t - 15)$. And Bob has travelled a distance $1.5t$. We have

$$(1)(t - 15) + 1.5t = 30,$$

and therefore $2.5t = 45$, so $t = 18$. Thus Al is already 3 seconds (metres) into his return trip; Al and Bob first meet 3 metres from the far end of the pool.

For the second time that they meet, again let it be after t seconds. Then Al has travelled a distance $30 + (t - 15)$. And Bob has travelled $1.5t$. These distance are equal, so $30 + (t - 15) = 1.5t$. This simplifies to $15 = 0.5t$, giving $t = 30$. That’s 15 seconds (metres) into Al’s second lap, so Al and Bob meet for the second time at the midpoint of the pool.

Once we know that they first meet 3 metres from the far end, we can solve the problem without further algebra. For Bob needs to travel 2 seconds to complete his first lap. In that time Al has travelled 2 metres, so is 5 metres ahead. Bob now gains half a metre a second, so needs 10 seconds to catch Al. That puts him midway in the pool..

Another way: There is no need at all of formal algebra—some thinking is enough. Let’s work backwards from the end. Bob took 40 seconds, Al took 45. So Bob won by 5 seconds, and at that time Al was 5 metres from finishing. Now run the movie backwards, so Al is swimming backwards at 1 metre per second, Bob at 1.5 metres per second, and Al has a 5 metre “lead.” Bob gains half a metre a second, so in 10 seconds has caught up. That’s 15 metres from the end, at the middle of the pool.

We can find the first meeting point in the same informal way. When Al finishes his first lap (15 seconds), Bob has travelled $(15)(1.5)$, that is, 22.5, and therefore the two are 7.5 metres apart. Now they are travelling *towards* each other, and the gap between them is closing at $1 + 1.5$ metres per second. So it takes 3 seconds for them to meet after Al’s turn.

Another way: In the picture, the horizontal axis represents time, and the vertical axis represents distance, as measured from the start of the race. The heavy line represents Al’s space-time path, while the dashed heavy line represents Bob’s. It is easy to draw these paths. Pick any convenient time scale, and any convenient distance scale. Al reaches the far end at time 15; his space-time position is then $(15, 30)$. Draw the line segment

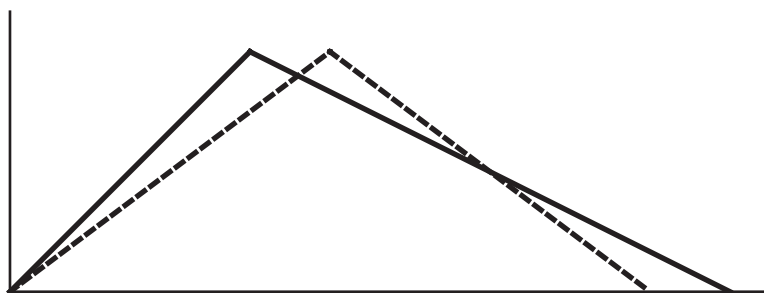


Figure 1: Al and Bob

joining $(0,0)$ to $(15,30)$. In another 30 seconds Al is finished, at position $(45,0)$. Draw the line segment joining $(15,30)$ to $(0,45)$. We now have Al's space-time path. Bob's is drawn in a similar way.

The picture shows that the two space-time paths intersect twice. If we have drawn the picture carefully we can measure where in space-time they meet. Or else now that we have a purely geometric problem we can use other ideas from geometry to do the calculation.