UBC Workshop Solutions A

Most students will have time to tackle only a few of these problems. Give them guidance about the ones to work on.

1. Alicia writes down her name over and over like this:

AliciaAliciaAliciaAliciaAliciaAliciaAliciaAliciaAliciaAlic. . . .

What is the 99th letter that she writes down? What is the 999th?

Solution. The name "Alicia" has 6 letters. Thus the sixth letter Alicia writes is an "a" (small). So is the 12th, 18th, 24th, and so on. Continue. The 60th letter is an "a," and so on, the 90th is, the 96th. Now we are close, the 97th is "A," the 98th is "l," the 99th is "i."

Since $900 = 150 \times 6$, the 900th letter that Alicia writes down is "a." And now things start all over again, she still needs to write 99 letters, and the last letter that she writes is "i."

Comment: The use of calculators for most computations has been seeping into the elementary schools. Some students may be interested in knowing how to use a simple calculator to find the remainder when a is divided by m. For example, to find the remainder when 6789 is divided by 47, divide on the calculator as usual, obtaining something like 144.44681, subtract the integer part 144, then multiply by 47. One obtains either 21 or (because of roundoff error) something very close to 21.

2. Erin, Lynn, and Tina bake one apple pie and one cherry pie. Erin eats one-half of the apple pie and one-quarter of the cherry pie. Next, Lynn takes one-half of what is left of the apple pie and one-third of what is left of the cherry pie. How much of each pie is left over for Tina?

Solution. This is perhaps too easy—the answers are one-quarter and one-half. But it is quite important to draw the pies. Why? Because we should treat the problem as a real problem, not as an exercise in fractions.

For the cherry pie in particular, the manipulation $(3/4)(1/3) = 1/4$, though technically sufficient, should be supplemented. It is easy to see from the meaning of "one-third" that one-third of what's left of the cherry pie is one-quarter of that pie. Some students will want to divide each of the (three) left over quarters of the cherry pie into thirds, that is, into pieces each one-twelfth of the original pie. If we take one such piece from each of the remaining quarters, we can reassemble the pieces to make one-quarter of a pie—precisely the meaning of $(3/4)(1/3) = 1/4$.

3. A 4 cm by 4 cm by 4 cm wooden cube is painted red all over. It is then cut up into $1 \times 1 \times 1$ cubes. How many of the little cubes are there? How many have one red face? Two red faces? Three red faces? No red faces?

Solution. This is an accessible problem, but many students will need help in starting. And a good start always is to make the problem concrete by drawing a picture. Many will draw an elaborately detailed picture, because this they know how to do.

One can also make a cube from paper, or find a small box in the classroom to serve as a pretend cube. Perhaps better than a box are four equal-sized books, which can represent the four layers of cubelets.

There are 64 cubelets, most easily counted as 4 layers of 16 each. Or else we can use a volume argument. The original cube has volume 4^3, each cubelet has volume 1^3, so there are $4^3/1^3$ cubelets. Note that the volume argument is far more abstract than the counting argument.

Each face of the original cube yields 4 cubelets wtih one red face, for a total of 6×4.

The 8 cubelets with three red faces come from the 8 corners of the original cube.

The cubelets with two red faces are trickier to count. We can think of them as coming 2 from each of the 12 edges, for a total of 12×2. Harder (too much harder?) is to see that there are 8 coming from each face, but then we have counted each of them twice, so the correct number is $(6 \times 8)/2$.

Note that we are here getting close to ideas that are useful in the analysis of regular polyhedra. Suppose that a regular polyhedron has E edges and F faces, and that each face has k edges. Then $kF = 2E$. (When we multiply the number of edges per face by the number of faces, we have counted each edge twice).

The cubelets with no red faces are the ones that come from the "inside" $2 \times 2 \times 2$ cube, so there are 8.

Note that we either get a check $(24 + 8 + 24 + 8 = 64)$ or can count all but one of the types directly and the last indirectly by using the fact that the sum is 64.

For anyone interested, it may be good to raise the question of the $n \times n \times n$ cube for general n, or at least for n quite a bit larger than 4, like 10.

The number with one red face is $6(n-2)^2$, the number with two is $12(n-2)$, the number with three is 8, and the number with none is $(n-2)^3$. Something interesting (but maybe not for grade 7!) is the following pretty expansion:

$$n^3 = ((n-2)+2)^3 = (n-2)^3 + 6(n-2)^2 + 12(n-2) + 8.$$

Note that the four terms in the expansion are the number of cubelets with 0 red faces, 1 red face, 2 red faces, and 3 red faces.

One can generalize further in various ways. For example we could take an $(a+2) \times (b+2) \times (c+2)$ box where a, b, and c are non-negative integers, paint it, cut it into $1 \times 1 \times 1$ cubes, and ask the same questions.

4. What fraction is halfway between $\dfrac{1}{4}$ and $\dfrac{1}{3}$?

Solution. It is reasonable to think of both fractions in terms of 12ths, so we are looking at three 12ths and four 12ths. Halfway in between is three and a half 12ths. Good enough, in a way, but we might want to express it as $7/24$.

Another way: Students have enough experience computing averages that they are aware, at some level, that $(a+b)/2$ is halfway between a and b. Thus an alternate approach is to immediately compute $(1/4 + 1/3)/2$.

Another way: Pictures are always nice. We can draw a number line, ask how far is it from $1/4$ to $1/3$ (or if I have a quarter-pound of gold and you have one-third of a pound how much richer are you). The distance is $1/12$. So to get halfway in between $1/4$ and $1/3$ we need to add $1/24$ to $1/4$.

5. A sports league has two conferences, East and West. Each conference has 10 teams. Every year, each team plays every team in its conference twice and plays every team in the other conference once. What is the total number of games played in the league during the year?

Solution. First look at the total number of games between teams of the Western Conference. Since each team plays every team in its conference twice, we can think of these two games as being "home" and "away."

Call the teams of the Western Conference W_1, W_2, W_3, …, W_{10}. Looks nice on a team sweater. Let's count the number of *home* games between Western Conference teams. Since any game is a home game for one of the teams, we will have counted all games between Western Conference teams. Each team, like W_1, plays 9 home games against Western Conference Teams. But there are 10 teams, so the number of games is 90.

By symmetry there are 90 games between Eastern Conference Teams.

Now we need to count the number of inter-conference games. Team W_1 plays 10 such games, as does W_2, and so on down to W_{10}, a total of 100 games. But this is *all* the inter-conference games.

Thus the total number of games is $90 + 90 + 100$.

Suppose more generally that the Western Conference has m teams and the Eastern Conference has n teams. The same argument shows that the total number of games is

$$m(m-1)+n(n-1)+mn.$$

More generally suppose that any two teams in the Western Conference play a games with each other, any two teams in the Eastern Conference play b games with each other, and any two teams in different conferences play c games. Then the total number of games is

$$\frac{am(m-1)}{2}+\frac{bn(n-1)}{2}+cmn.$$

Another way: It is informative to view things more geometrically. To count the number of games between Western Conference teams, draw a 10×10 array with rows and columns labelled with the team names. The two games between every pair of teams are represented by the 90 entries in the array (100 minus the 10 "diagonal" entries). If there were only one game for each pair of teams, the games could be represented for example by the 45 entries in the triangle above the main diagonal.

Representing the inter-conference games is even simpler: label the rows of a 10×10 array with the Western Conference team names, and the columns by the Eastern Conference team names.

Another way: Here is an argument that is at once simpler and harder. Look at a particular team, say W_1. This team plays 2 games against each of the 9 Western Conference teams (18 games) and 1 each against the 10 Eastern Conference teams, for a total of 28 games. Each team in the league does the same thing, for a total of (28)(20). Well, not really. When we multiplied 28 and 20, we were counting twice each game that team W_1 played, once from W_1's point of view and once from the other team's point of view. So the total number of games is (28)(20)/2. Note that this approach uses the symmetry between the two conferences.

6. An arena has 21000 seats. It is divided into four sections. Section A has twice as many seats as Section B. Section C has twice as many seats as Section D. Section B has 1000 more seats than Section D. How many seats are in each section?

Solution. For people who are comfortable with basic algebra, it is natural to let x be something or other, maybe the number of seats in D—that way we can avoid subtraction and fractions for a while.

So C has $2x$ seats, B has $x+1000$, A has $2x+2000$. Add up. We get $6x+3000=21000$, so $x=3000$ and we are finished.

Or else we could start with four letters a, b, c and d, write down the obvious equations and simplify. But at this stage students are not comfortable with several variables. This discomfort has long historical roots. Diophantus and early Islamic mathematicians such as al-Khwārizmī dealt with some degree of comfort with a single "unknown." But many hundreds of years elapsed before mathematicians handled several variables with confidence.

Another way: We could also probably guess and refine. This works fast since the answer is too simple. Maybe guess that D has 0 seats? Then C also has 0, B has 1000, A has 2000, for a total of 3000, not enough. Maybe now guess 1000 for D. That gives arena size 9000. In fact adding 1 person to our guess for D adds 2 to C, 1 to B, 2 to A, so adds 6 people. We want to get from 3000 to 21000, need to add 6 people 3000 times, giving 3000 people for section D. (A very good guess for the size of D is -500.)

Another way: Or else first deal with the 1000 "extra" seats that B has, and therefore the 2000 extra that A has. Remove these, we have 18000. And now A and B clearly balance C and D, so C and D have 9000, and therefore D has 3000.

There are other ways of reasoning the problem through. A picture can be useful.

7. Lisa's bag of groceries costs $19.53. She pays with a $20 bill. The cashier has many quarters, nickels, and dimes in his till, but only 9 pennies. How many different combinations of coins can Lisa get as change?

Solution. We first find out how much money she gets back. It is unpleasant to do a formal subtraction—there is too much "borrowing."

There are a couple of easy ways around it. If she had paid with a $19.99 bill, the subtraction would be easy, we would get 0.46. But she used a $20 dollar bill, so she gets one cent more.

Or else we can do things more or less like cashiers used to do, 19.53 to 19.60 is 7, and to 20 another 40.

With 47 cents, it is not worthwhile to develop general theory, but it is probably worthwhile to count in some systematic way. For example, we can count all the patterns in which we use 1 quarter. Then we can count all the patterns that use 0 quarters. We will write a * for each way we find.

If we use 1 quarter, then we need to make up 22 cents. Look at the number of dimes: it is 2, 1, or 0. If we use 2 dimes, we need to use 2 pennies (*). If we use one dime, we need 1 nickel or 2, with the rest pennies (**). If 0 dimes, then 4 nickels or 3, with the rest pennies (**). The total number of ways that use 1 quarter is 5.

If we use 0 quarters, then the number of dimes is 4, 3, 2, 1, or 0. If 4 dimes, we have 1 nickel and the rest pennies, or 7 pennies (**). If 3 dimes, again (**) and the same for 2, 1, or 0, for a total of 10.

So there are $5 + 10$ ways to give change.

Another way: The counting can be organized in other ways. The only important thing is that it be *organized,* so that we will get the right answer and *know* that we got it, and be able to attack larger problems.

We use either 2 pennies (then need 45 cents) or 7 pennies (then need 40 cents). How can we make 45 cents? Either we use no nickels (so 2 dimes and a quarter, or we use at least one nickel, but then we are adding a nickel to 40 cents. So there is *one more way* of making 45 cents from nickels, dimes, and quarters than there is of making 40 cents.

Let's count the ways to make 40 cents. If we use 1 quarter, everything is determined by the number of dimes (0 or 1). If we use 0 quarters, then there are 5 choices (0 to 4) for the number of dimes. So there are 7 ways to make 40 cents, hence 8 ways to make 45, and by the remarks above the total is $7 + 8$.

8. Three runners compete in a 100 meter race. How many possible orders of finish are there, if ties are allowed?

Solution. The runners are of course called A, B, and C. Now we must patiently list all the possible orders of finish. There are not many.

Count first the orders of finish with no ties. Any one of the 3 runners could be first. Let's count how many orders of finish there are with say A first: clearly there are 2. And there are 2 with B first, 2 with C first, for a total of 6.

Now count the number of orders of finish with exactly 2 runners tied. They could be (i) tied for first or (ii) tied for last.

To count the orders of finish with two people tied for first, think of the last runner. She can be chosen in 3 ways. Once that is done, we are finished. Similarly, there are 3 orders of finish with exactly 2 runners tied for last, for a total of 6.

And of course there is the possibility of a triple tie. Thus overall there are $6 + 6 + 1$ possible orders of finish.

Comment: If someone finds three runners too simple, perhaps she can look at four runners (the answer is 75).

9. A poster is 40 centimeters wide. There are two pictures on the poster. Each picture is 25 cm wide and 20 cm high. Together the pictures take up one-third of the area of the poster. How many centimeters are in the height of the poster?

Solution. The area taken up by each picture is 25×20, namely 500 (square centimetres), so the two pictures together have area 1000. This is one-third of the area of the poster, so the poster has area 3000. It has width 40, so it has height 3000/40, that is, 75. Students can be guided to do mental arithmetic: $3000/40 = 300/4 = 150/2 = 75$.

10. Amy, Mark, and Suzy together can just manage to pull a 140 kilogram sled. Mark can pull twice as much weight as Suzy, and Amy can pull twice as much weight as Mark. Up to how much weight can Suzy pull all by herself?

Solution. This is a simple algebra problem: let x be the weight Suzy can pull, then Mark can pull $2x$, and Amy $4x$, for a total of $7x$ where $7x = 140$. Thus $x = 20$.

Another way: Many students will not have seen variables yet. The idea of variable can be simulated by drawing a box to represent the weight that Suzy can pull. Then draw the 2 boxes Mark can pull, the 4 boxes that Amy can pull. So pulling all together they can pull 7 boxes, so 7 boxes weigh 140, and we are finished.

Another way: It is not unreasonable to do some sort of "guess and check." It would be nice, however, if after one unsuccessful guess we can mull over the results and go directly to the right answer.

A "silly" guess turns out to be useful. Guess that Suzy can pull 1 kilogram. Then altogether they can pull 7. Now the scaling up to 140 is reasonably clear. Guessing that Suzy can pull 50 is less useful, for then they can altogether pull 350, and multiplying by the factor 140/350 will not seem obvious to many.

11. If Ken gives Joey one of his quarters, they will have the same number of quarters. If Joey gives Ken one of her quarters, Ken will have twice as many quarters as Joey has. How many quarters do they each have?

Solution. The answer (Ken 7, Joey 5) can be found by trial and error. Or by algebra. Or by some sort of reasoning procedure.

It is clear that Ken started with 2 more quarters than Joey, since when Ken handed over a quarter the fortunes were even. And Joey has at least one quarter. So what Joey has can be thought of as a bunch, plus 1 quarter.

And what Ken has can be thought of as a bunch, plus 1 quarter, plus 2 quarters, that is, a bunch plus 3 quarters. When Joey hands a quarter to Ken, Ken has the bunch plus 4 quarter, and Joey has the bunch. But Ken has twice as much as Joey, so the bunch has 4 quarters in it, we are now at Joey 4, Ken 8, so we started with 5, 7.

12. How many positive integers are factors of 720? Here are a few of them: 1, 5, 8, 360, 720.

Solution. It turns out that there are 30 of them, so if we are going to make a list it should be an efficient list. Maybe we should first factor 720:

$$720 = 2^4 \times 3^2 \times 5.$$

First look at the factors of 720 with no 2's in them, that is, factors of $3^2 \times 5$. These perhaps can be listed explicitly: 1, 3, 5, 9, 15, 45, six of them

Then look at the factors of 720 that have exactly one 2 in them. These are all numbers which are twice numbers in our first list, explicitly 2, 6, 10, 18, 30, 90, but we don't need to list them to see there are six.

Then look at the factors of 720 that have exactly two 2's in them. These are all numbers which are twice a number in the preceding list, or four times a number in the first list, in all six numbers.

Then look at the factors of 720 that have three 2's. There are six. And there are six that have four 2's. So the total is $6 + \cdots + 6$ (five sixes, 30).

Another idea, a bit unpleasant but not too bad, is to note that since $27 \times 27 = 729$, whenever we express 720 as $a \times b$, the smaller of a or b must be less than 27, and the bigger must be bigger than 27. So the factors of 720 come in pairs (a, b) where a is less than 27 and $b = 720/a$. Now start listing all the factors less than 27: 1, 2, 3, 4, 5, 6, 8, 9 ,10, 12, 15, 16, 18, 20, 24, fifteen in all. Then there are their fifteen mates, for a total of 30.

Or else we start from $720 = 2^4 \times 3^2 \times 5$, and make up a divisor of 720 by first deciding how many 2's we want (we can have 0 to 4 of them, 5 choices). For every choice of how many 2's, we have 3 choices for how many 3's (0, 1, or 2), and 2 choices for how many 5's (0 or 1). The total number of choices is thus 5×2.

It may be interesting to some students how one can find the number of positive integer divisors of n where

$$n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$$

where the p_i are distinct primes. The same argument as the one above shows that there are $(e_1 + 1)(e_2 + 1) \cdots (e_k + 1)$ divisors.

13. The plane was full when it left Vancouver. In Seattle, half the people got off and 28 got on. In Portland, half the people got off, 40 got on, and the plane was full again. How many people were on the plane when it left Vancouver?

Solution. It is not unreasonable to find our way by experimenting. Hard to know what guess we would start with. Say 40. The 48 people leave Seattle, and therefore 64 leave Portland. Is 40 too big or too small? Maybe it is not obvious.

Try for what next? Maybe 41? So $41/2 + 28$ leave Seattle. Possible but perhaps a bit gruesome. And 42 gives us the same problem of half a person when we leave Portland. And of course 43 is no good. Try 44.

So 50 leave Seattle, 65 leave Portland. Still not good, but 44 is closer to 65 than 40 was to 64. Maybe try for 48 out of Vancouver. Then get 66 out of Portland. Interesting pattern! We add 4 to the number leaving Vancouver, and the number out of Portland rises only by 1, so the "gap" went down by 3.

With this observation we can finish things. At 40 the gap between Vancouver and Portland was 24. Going up by 4 in Vancouver closes the gap by 3. We nedd eight 3's to get rid of the gap of 24. So in Vancouver we should go up by 8 4's, that is by 32, up to 72. Check. It works.

Another way: A good way to handle things is with proto-algebra. Maybe represent the number of people leaving Vancouver by a line. Then out of Seattle come half the line plus 28. Out of Portland come half of half the line plus 14 plus 40, that is, a quarter of the line plus 54. And this is the whole line. Then we can do arithmetic, but maybe it is better, since one-quarter of the line comes into the game, to think of the people leaving Vancouver (the original line segment) as being made up of 4 equal line segments. After a while we get that 3 of the chunks add up to 54 so one chunk is 18 and 4 chunks give 72.

Another way: Of course we could go through the x business. So there are x when we leave Vancouver, then $x/2+28$ leave Seattle, and $1/2(x/2+28)+40$ leave Portland. But this is x. So

$$1/2(x/2+28)+40=x.$$

One way to solve this is to multiply through. We get

$$x/4+14+40=x$$

so $3x/4=54$, $x=72$.

But fractions are a tricky business, so it may be better to unravel our basic equation. We get $1/2(x/2+28)=x-40$ so $x/2+28=2x-80$, so $x/2=2x-108$ so $x=4x-216$ so $3x=216$ and $x=216/33=72$.

14. Ten consecutive odd integers add up to 800. What is the smallest of these integers? An example of 10 consecutive odd integers is 7, 9, 11, 13, 15, 17, 19, 21, 23, 25—but they don't add up to 800.

Solution. Here is an ugly solution. Let the smallest number be x. Let's not worry too much about whether it is odd or not. Then the others are $x+2$, $x+4$, $\dots$, $x+18$. Add up, we get $10x+90$, which is 800, giving $x=71$. Since 71 is an odd integer, 71, $71+2$, $71+4$, $\dots$, $71+18$ are consecutive odd integers, and they have the right sum, so the answer is 71.

Comment: The last sentence is not quite superfluous. Let us change the problem slightly to "Ten consecutive odd integers add up to 790. What is the smallest of these integers?” A calculation almost identical to the one above gives $x = 70$, which is clearly wrong, 70 is not odd. In general when we write “Let $x = \dots$” we are making the implicit assumption that an object with the desired properties exists. If this is not true, conclusions that we draw about x will be wrong.

Another way: Here is something a little more attractive. Look at the two middle numbers, and let x be the number halfway between them. So the numbers are $x-9$, $x-7$, $x-3$, $x-1$, $x+1$, $x+3$, $x+5$, $x+7$. Add up, grouping in the obvious way. We get $10x = 800$, $x = 80$, so our numbers are all odd and the smallest is $80-9$.

Another way: Or else “guess” that the numbers are 1, 3, 5, 7, 9, 11, 13, 15, 17, 19. Add up, we get 100, bad guess, the sum is off by 700. So add 70 to each number, the sum is right!

Another way: A cute (maybe too cute) idea is to “guess” that the numbers are -9, -7, -5, -3, -1, 1, 3, 5, 7, 9. Spectacularly bad guess, but the sum is easy to find, it is 0. If we add 80 to each number we get the right sum.

Another way: Or else we can start from the sequence 80, 80, $\dots$, 80 (ten 80’s), which has the right sum but does not consist of consecutive odd integers. We modifiy this sequence so as to keep the sum unchanged while satisfying the “consecutive odd” condition. Change the two 80’s closest to the middle to 79 and 81. So now we are at

$$80, \quad 80, \quad 80, \quad 80, \quad 79, \quad 81, \quad 80, \quad 80, \quad 80, \quad 80.$$

Change the two 80’s closest to the middle to 77 and 83, then the two remaining 80’s closest to the middle to 75 and 85, and so on until we get to the sequence 71, 73, 75, 77, 79, 81, 83, 85, 87, 89.

Comment: It is worthwhile to think about generalizing. Let’s stick to consecutive odd numbers, but look for n consecutive odd numbers whose sum is S.

If there are such numbers, then n must be a factor of S. For if n is even we can without loss of generality take the consecutive odd numbers to be $a \pm 1$, $a \pm 3$, and so on. Their sum is an, a multiple of n. And if n is odd we can take the numbers to be a, $a \pm 2$, and so on, again with sum an.

Now assume that S/n is an integer. Can we automatically find n odd consecutive integers with sum equal to S? Not necessarily: the sum of an odd number of odd numbers is odd, so if n is odd we must have S odd. And if n is even, and we let the numbers be $a \pm 1$, $a \pm 3$, and so on, then $a = S/n$, and therefore we are forced to have S/n even.

Apart from these restrictions, there is no problem. If n is odd, and S is odd, and S/n is an integer (necessarily odd), we can use the numbers a,

$a \pm 2$, $a \pm 4$, and so on for a total of n numbers. And if n is even and S/n is even, let $a = S/n$ and use $a \pm 1$, $a \pm 3$, and so on for a total of n numbers. We can extend these observations to sums of consecutive members of an arithmetic progression of integers with common difference d (we have just dealt with $d = 2$).

By the way, the following problem has been discussed on the Secondary math teachers' listserv. Is it true that every positive integer which is not a power of 2 can be expressed as the sum of two or more consecutive positive integers? (Yes.) In how many ways?

15. A, B, C, and D are running a marathon along a straight road. As usual, A is in front, B is next, C is behind B, and D is behind C. At this instant, A is 1 mile ahead of C, B is 4 times as far from A as she is from C, and D is also 4 times as far from A as she is from C. What is the distance, in miles, between B and D?

Solution. If we look at the space between A and C (one mile) then maybe it is clear, if we draw a little picture, that C is one-fifth of a mile behind B, and B is four-fifth of a mile behind A.

Now look at the relationship between A, C, and D. So from A to D is four 'parts' while A to C is 3 parts. But A to C is one mile, so A to D is 1 mile plus 1/3 of a mile. Because we are dividing a mile into 5 parts, and also into 3 parts, it may be a good idea to break up the mile into 15 parts. The rest is easy. Draw a line to represent the 1 mile distance between A and C, and break it up into 15 equal parts. Then the rest can be figured out: B to D is 8 parts, eight-fifteenths of a mile.

We can also use "fractions." Fine of course, but a bit further from basic intuition.

16. The interior of cooking pot A is a cylinder with base diameter 15 cm and height 10 cm. The interior of cooking pot B is a cylinder with base diameter 30 cm and height 40 cm. Pot A is filled with water and the contents are poured into pot B. After this has been done a total of six times, how many cm deep is the water in pot B?

Solution. We can trot out the machinery of volumes. The volume of the small pot is $\pi(15/2)^2(10)$. If water is h centimetres deep in the big pot, then the volume of water is $\pi(30/2)^2h$. So to find how much one small potful contributes to height in the big pot, we set

$$\pi(15/2)^2(10) = \pi(30/2)^2h$$

and after some cancellation (or a calculator computation) find that $h = 10/4 = 2.5$. So 6 potfuls give depth of 15 centimetres.

Another way: The big cooking pot has diameter twice the diameter of the little one, so the area of the bottom of the big pot is 4 times the area of

Drop	L	Q	RD	BD	RN	BN
pick up N	no	no	no	no	yes	yes
pick up D	no	no	yes	yes	no	no

Table 1: The Dropped Coin

the bottom of the little one. That means that 10 cm of water in the little pot only give us 10/4 in the big pot. Repeat 6 times.

Note that everything would be intuitively obvious if pots had square bases—four potfuls from A give us 10 cm in pot B.

17. Three swimmers had a race across a small lake. Each swam at constant speed. When A finished, she was 20 metres ahead of B, and 40 metres ahead of C. When B finished, she was 20.5 metres ahead of C. Over how many metres was the race?

Solution. When A finished, B had 20 metres to go, and had a 20 metre lead on C. While swimming 20 metres she built up her lead by 0.5 metres.

But B had built a lead of 20 metres over C. Since she increases the lead by 0.5 metres for every 20 metres she swims, or by 1 metre for every 40, she needs 40×20 to build a lead of 20 metres. That gets B to 20 metres from the end, so the length of the race is 820 metres.

18. Suzie has \$1.55—1 loonie, 1 quarter, 2 dimes, and 2 nickels. She sees a dime and a nickel on the ground and picks one of them up. While picking up her new coin, she drops one of her old coins. What is the probability that she still has \$1.55?

Solution. This is not easy, particularly since many of the students have not seen a probablity argument. With such students, it is best to not do the problem.

To solve the problem, one encourages people to draw up a 2×6 table. She picked up either a nickel or a dime. The first row deals with picking up a nickel, and shows the various coins she might drop; the second row does the same thing with picking up a dime. The dimes she has are labelled RD (red dime) and BD (blue dime) and we use similar notation for the nickels. We could just repeat the letters but for other problems distinguishing between coins may be useful. The table entries show whether or not she still has \$1.55, yes or no.

The 12 entries in the table are equally likely. Among them we have 4 "yes." So the probability that the result is "yes," meaning that she still has \$1.55, is 4/12.

Another way: We can also argue in a more streamlined way as follows. Whether she picks up the nickel or the dime, the probability that she drops the matching coin is 2/6, for 2 of her 6 coins match the coin she picked up. So the probability she still has $1.55 is 1/3.

This argument is correct. But the table approach that we used first works in many other situations, for example if there was 1 nickel and 2 dimes on the ground, or if among her old coins there was an unequal number of nickels and dimes.

19. A 20 cm by 20 cm square cake is 7 cm high. It has smooth icing on top and apricot glaze on the sides. Divide the cake among 5 people so that cake, icing, and glaze are all divided equally. Don't use a potato masher. Hint: The midpoint of the top of the cake is involved.

Solution. A picture of the top of the cake is helpful.

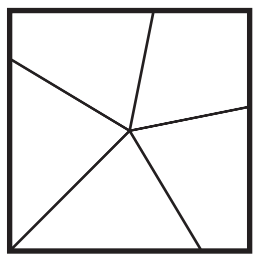

Figure 1: Dividing a Cake Fairly

We will use a knife and cut the cake in a more or less conventional way, straight down. If everyone gets the same *area* of icing, then since cuts are straight down, everyone will get the same *volume* of cake. That leaves a lot of freedom. But to make sure everyone gets the same amount of glaze, we need to make sure that they all get one-fifth of the *circumference* of the cake. The circumference is 80, so let's try to give everyone 16 cm of circumference, say as shown in the picture. So start at any point on the circumference, make a straight cut to the center. Then travel 16 cm around the cake, make another cut, and so on.

We need to verify that everyone gets the same area of the top of the cake. The fact that this is so comes from the fact that the area of a triangle is half the base times the height. All the triangular pieces shown in the picture have the same base (16) and the same height (10). The non-triangular pieces, that is, the pieces that go around a corner of the cake, can be divided into 2 triangles, with bases that add up to 16 and with height 10, so again the right area.

20. If n is a positive integer, then $n!$ (read this as "n factorial," or "factorial n") is the product of all the numbers from n down to 1. For example,

$$4! = 4 \times 3 \times 2 \times 1 = 24, \qquad 5! = 5 \times 4 \times 3 \times 2 \times 1 = 120.$$

Find the highest power of 2 that divides 32!. Note that for example the highest power of 2 that divides 5! is 2^3.

Solution. We have

$$32! = 1 \times 2 \times 3 \times 4 \times 5 \times \cdots \times 30 \times 31 \times 32.$$

We can see how many 2's there are "in" each of 1, 2, 3, 4, 5, …, 32, and add up.

There are no 2's "in" 1, 3, 5, 7, …, 31. The number 2 has one two in it (1), the number 4 has two 2's in it (2), the number 6 has one (1), the number 8 has three (3), the number 10 has one (1), the number 12 has two (2), and so on. Add up. It is not too bad.

We can save quite a few steps by noting that the numbers 2, 6, 10, 14, 18, 22, 26, 30 each have one 2, for a total of 8. And 4, 12, 20, 28 have two, for a total of 8. And 8, 24 have three, for a total of 6. And 16 has 4, and 32 has 5. Add; we end up with 31. So the highest power of 2 that divides 32! is 2^{31}.

Another way: There is a much nicer way of looking at things. Imagine that each of the numbers from 1 to 32 has to pay a one-dollar tax for each 2 in it. So for example the number 8 pays a tax of $3.

Collect one dollar from each of the 32/2 even numbers from 1 to 32. Some numbers, like 4, 8, and so on still owe tax. Collect a dollar from each of the 32/4 multiples of 4 from 1 to 32. The numbers 8, 16, and so on still owe money. Collect a dollar from each of the 32/8 multiples of 8, a dollar from each of the 32/16 multiples of 16, and finally a dollar from each of the 32/32 multiples of 32. Now everyone has paid the proper tax, which adds up to

$$32/2 + 32/4 + 32/8 + 32/16 + 32/32$$

that is 31.

Let p be a prime. A mild modification of the idea above enables us to calculate the highest power of p that divides $n!$.