UBC Workshop Solutions C

1. Let $P = (5, 0)$, $Q = (4, 4)$, and $R = (0, 5)$. Find the area of $\triangle PQR$.

Solution. Almost anything works. For example, let $O = (0, 0)$. Quadrilateral $OPQR$ is split into two equal triangles by the line OQ. Triangle OPQ has base OP of length 5, and height 4, so it has area 10. Triangle ORQ also has area 10, so $OPQR$ has area 20.

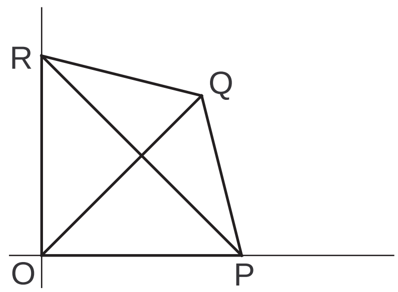

Figure 1: The Area of $\triangle PQR$

But $OPQR$ can be dissected into $\triangle OPR$, with area $25/2$, and $\triangle PQR$. It follows that $\triangle PQR$ has area $20 - 25/2$, that is, $15/2$.

Another way: Let M be the midpoint of PR. Then $\triangle PQR$ can be viewed as a triangle with base PR and height MQ. It is easy to verify that $OQ = 4\sqrt{2}$ while $OM = 5\sqrt{2}/2$. Thus $MQ = 3\sqrt{2}/2$. Since $PR = 5\sqrt{2}$, the area of $\triangle PQR$ is $(1/2)(3\sqrt{2}/2)(5\sqrt{2})$, that is, $15/2$.

Another way: Let M be the midpoint of PR. Then M has coordinates $(5/2, 5/2)$. The area of $\triangle MPQ$ is equal to the area of $\triangle OPQ$, namely $(1/2)(5)(4)$, minus the area of triangle OPM, namely $(1/2)(5)(5/2)$. Thus $\triangle MPQ$ has area $15/4$. Double this to find the area of $\triangle PQR$.

Another way: Instead of taking advantage of special symmetries in this problem, we can use more general purpose tools. Drop a perpendicular from Q to the x-axis, and let its foot land at X. Then $X = (4, 0)$. Trapezoid $OXQR$ has area $(1/2)(5 + 4)(4)$, namely 18, $\triangle XPQ$ has area 2, and $\triangle OPR$ has area $25/2$, and therefore $\triangle PQR$ has area $18 + 2 - 25/2$.

The same idea can be used to show the following result. Let $A = (x_1, y_1)$, $B = (x_2, y_2)$, and $C = (x_3, y_3)$. Then the area of $\triangle ABC$ is

$$\frac{1}{2}|x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3|.$$

(This somewhat mysterious-looking formula makes more structural sense from a vector point of view.)

Another way: Let $\triangle ABC$ have sides of length a, b, and c, and let $s = (a+b+c)/2$, so s is the semi-perimeter. Then by *Heron's Formula* the area of $\triangle ABC$ is equal to

$$\sqrt{s(s-a)(s-b)(s-c)}.$$

It is easy to see that the sides of $\triangle PQR$ are $\sqrt{17}$, $\sqrt{17}$, and $5\sqrt{2}$. Substituting directly into Heron's Formula, we find that the area is

$$\sqrt{(\sqrt{17}+(5/2)\sqrt{2})((5/2)\sqrt{2})((5/2)\sqrt{2})(\sqrt{17}-(5/2)\sqrt{2})}.$$

This simplifies to $15/2$.

2. A 10 metre by 16 metre pool is surrounded by a walkway of uniform width whose area is 87 square metres. How wide is the walkway?

Solution. This should be routine, since it exercises algebraic skills that will be needed in calculus courses. Sketch the pool and its walkway. Let the

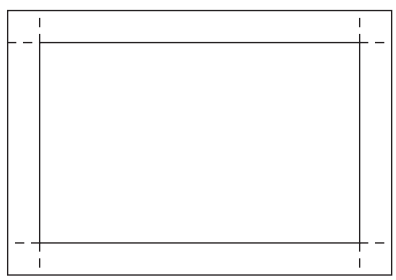

Figure 2: Pool and Walkway

width of the walkway be w. The walkway consists of four $w \times w$ squares (in the corners) with total area $4w^2$, and two $w \times 16$ rectangles, and two $w \times 10$ rectangles. The combined area of the parts we have divided the walkway into is $4w^2+52w$, so we arrive at the equation $4w^2+52w-87=0$.

Alternately, pool plus walkway form a $(2w+10)\times(2w+16)$ rectangle. Subtract $(10)(16)$ from its area to get the area of the walkway. We conclude that

$$(2w+10)(2w+16)-(10)(16)=87$$

and again arrive at $4w^2+52w-87=0$. This quadratic equation can be solved as usual by the quadratic formula, or by completing the square—these are the "right" way to handle the problem. We should *not expect* factoring to yield an easy path to the solution. But the builders of the walkway thoughtfully arranged for the quadratic to factor. We have

$$4w^2 + 52w - 87 = (2w + 29)(2w - 3)$$

so the walkway has width 3/2.

3. A sports league has two conferences, East and West. Each conference has 10 teams. Every year, each team plays every team in its conference twice and plays every team in the other conference once. What is the total number of games played in the league during the year?

Solution. First look at the total number of games between teams of the Western Conference. Since each team plays every team in its conference twice, we can think of these two games as being "home" and "away."

Call the teams of the Western Conference W_1, W_2, W_3, …, W_{10}. Looks nice on a team sweater. Let's count the number of *home* games between Western Conference teams. Since any game is a home game for one of the teams, we will have counted all games between Western Conference teams. Each team, like W_1, plays 9 home games against Western Conference Teams. But there are 10 teams, so the number of games is 90.

By symmetry there are 90 games between Eastern Conference Teams.

Comment: Many students will know how many ways there are to "choose" two teams from 10, so they will compute this number (45) and double. In a sense this is not quite reasonable: it is really *easier* to count the games if there are two between each pair of teams. If there is only one we can divide by 2.

Now we need to count the number of inter-conference games. Team W_1 plays 10 such games, as does W_2, and so on down to W_{10}, a total of 100 games. But this is *all* the inter-conference games.

Thus the total number of games is $90 + 90 + 100$.

Suppose more generally that the Western Conference has m teams and the Eastern Conference has n teams. The same argument shows that the total number of games is

$$m(m - 1) + n(n - 1) + mn.$$

More generally suppose that any two teams in the Western Conference play a games with each other, any two teams in the Eastern Conference play b games with each other, and any two teams in different conferences play c games. Then the total number of games is

$$\frac{am(m-1)}{2} + \frac{bn(n-1)}{2} + cmn.$$

Another way: It is informative to view things more geometrically. To count the number of games between Western Conference teams, draw a 10×10 array with rows and columns labelled with the team names. The two games between every pair of teams are represented by the 90 entries in the array (100 minus the 10 "diagonal" entries). If there were only one game for each pair of teams, the games could be represented for example by the 45 entries in the triangle above the main diagonal.

Representing the inter-conference games is even simpler: label the rows of a 10×10 array with the Western Conference team names, and the columns by the Eastern Conference team names.

Another way: Here is an argument that is at once simpler and harder. Look at a particular team, say W_1. This team plays 2 games against each of the 9 Western Conference teams (18 games) and 1 each against the 10 Eastern Conference teams, for a total of 28 games. Each team in the league does the same thing, for a total of (28)(20). Well, not really. When we multiplied 28 and 20, we were counting twice each game that team W_1 played, once from W_1's point of view and once from the other team's point of view. So the total number of games is $(28)(20)/2$. Note that this approach uses the symmetry between the two conferences.

4. A paper drinking cup is cone-shaped. When there is water in the cup to a depth of 4 inches, the cup contains 16 cubic inches of water. How many cubic inches of water are in the cup when the water is 3 inches deep?

Solution. It is tempting to first reach for a formula, and then for a calculator. If a cone has height h and 'base' radius x, then its volume V is given by

$$V = \frac{\pi x^2 h}{3}.$$

It would be easy to compute the volume of water when the depth is 3 inches if we knew the base radius r. We don't know r, but we can compute the base radius R when the depth of water is 4 inches.

When the water cone has height 4, our cup contains 16 cubic inches. It follows that $16 = 4\pi R^2/3$, and therefore

$$R = \sqrt{\frac{(16)(3)}{4\pi}}.$$

Comment: At this point many would reflexively reach out for the calculator and compute R to some number of decimal places. This is a *very bad* habit. Our expression for R has *structure.* Pushing it through a calculator turns a structured object into a jumble of digits.

The two water cones are *similar.* It follows that

$$\frac{r}{R} = \frac{3}{4}$$

and therefore $r = (3/4)\sqrt{(16)(3)/(4\pi)}$.

Now we can compute the volume of water, namely $3\pi r^2/3$. If we use the value of r found above, we get quickly that the volume is $(3)(3/4)^2(16)/4$ (the π's cancel). Now, with calculator or without, we find that the volume is 6.75 cubic inches.

Another way: There is a *much* better way of looking at the problem—so much better that the solution we have just given should be called the wrong solution.

The 'small' water cone is just a scaled down version of the big water cone. The *linear* scaling factor is 3/4, that is, all lengths get multiplied by 3/4. If we multiply the dimensions of an object by the linear scaling factor t, then areas scale by the factor t^2, and volumes scale by t^3. So the volume of our small water cone is $(3/4)^3(16)$.

5. Find all (real) values of k such that $x^2 - 2kx + k + 1 = 0$ has no real roots.

Solution. Complete the square. We have

$$x^2 - 2kx + k + 1 = (x - k)^2 - k^2 + k + 1,$$

so our original equation can be rewritten as

$$(x - k)^2 = k^2 - k - 1.$$

For any k, by suitably choosing x we can make $(x - k)^2$ equal to any preassigned non-negative quantity. So our original equation has no solution if and only if $k^2 - k - 1 < 0$.

Look at the curve with equation $y = k^2 - k - 1$, where k is a variable. This curve is an upward-facing parabola. So y is only negative between the two roots of the equation $k^2 - k - 1 = 0$. It follows that our original equation has no real roots if and only if

$$\frac{1 - \sqrt{5}}{2} < k < \frac{1 + \sqrt{5}}{2}.$$

We can start in a slightly different way. All roots, real or not, of our equation are given by the formula

$$x = \frac{2k \pm \sqrt{(2k)^2 - 4(k + 1)}}{2} = k \pm \sqrt{k^2 - k - 1}.$$

So there are no real roots if and only if $k^2 - k - 1 < 0$. Now proceed as before.

6. A 6×4 (base 6, height 4) rectangle is divided into twenty-four 1×1 squares by drawing 3 lines parallel to the base of the rectangle and 5 lines perpendicular to its base. How many different rectangles can be formed using one or more of the 1×1 squares?

Solution. Since the numbers aren't large, we can probably manage the count without much theory. Draw a picture like Figure 3. For now pay no special attention to the two thicker lines. There are clearly 24 1×1 rectangles. Now count the 2×1 (base 2, height 1) rectangles. By keeping our eyes on the picture we can see that there are 5 in each row, for a total of 20. In the same way we find that there are 12 3×1 rectangles. We can continue in this way and after a while reach the answer.

But all this is somewhat tedious; let's count things in a more structured way. It is natural to organize the count by rectangle size. Start with

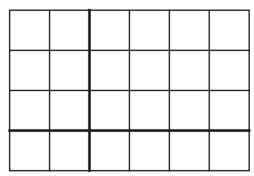

Figure 3: Counting Rectangles

"large" ones—there are fewer of them. There is 1 6×4 rectangle, 2 6×3, 3 6×2, and 4 6×1. Continue by counting the 5×4, the 5×3, and so on.

We show how to figure out for example the number of 4×3 rectangles (width four, height three). Start with a 4×3 wedged into the Northeast corner of the original rectangle, and see what freedom of movement it has: 0, 1, or 2 units to the West and/or 0 or 1 units South, for a total of $3 \cdot 2$ possibilities.

Another way of putting it is that a 4×3 is completely determined once we specify its Southwest corner. And the Southwest corner of a 4×3 has to lie in the Southwest rectangle determined by the two thick lines in Figure 3. The picture shows that there are $3 \cdot 2$ choices for that Southwest corner.

This kind of reasoning shows that there is 1 6×4 rectangle, 2 6×3 rectangles, 3 6×2, and 4 6×1, for a total of $1 + 2 + 3 + 4$.

Similarly, there are $2 \cdot 1$ 5×4 rectangles, $2 \cdot 2$ 5×3, $2 \cdot 3$ 5×2, and $2 \cdot 4$ 5×1, for a total of $2(1 + 2 + 3 + 4)$. Continue in this way. After a while we get that the total number of rectangles is

$$(1+2+3+4)+2(1+2+3+4)+3(1+2+3+4)+\cdots+6(1+2+3+4).$$

This number can be rewritten as $(1 + 2 + 3 + 4)(1 + 2 + 3 + 4 + 5 + 6)$. It turns out to be 210.

The idea generalizes. If (including the sides of the rectangle) there are m East–West lines and n North–South lines, the count proceeds in exactly the same way, with result

$$(1 + \cdots + (m-1))\,(1 + \cdots + (n-1))\,.$$

The above formula is easier to compute with if we use the fact that in general $1 + 2 + \cdots + k = k(k+1)/2$.

Another way: Instead of organizing the count by size, give it structure by counting for each possible P the number of rectangles that have P as Southwest corner. Start with P the Southwest corner of the whole rectangle. Why start there? We have a coordinate system in mind, and think of the Southwest corner as $(0,0)$.

To make a rectangle with Southwest corner $(0,0)$, we need to pick its Northeast corner Q. A look at Figure 3 shows that there are $6 \cdot 4$ ways of picking Q. Now move P one unit North. Why North? We are increasing the y-coordinate by 1. There are then $6 \cdot 3$ rectangles with P as Southwest corner. Go on like that, systematically. We get a total of $6(4+3+2+1)$ rectangles whose Southwest corner is on the y-axis.

Now start with $P = (1,0)$. There are $5 \cdot 4$ rectangles that have P as bottom left corner. Move to $(1,1)$, then $(1,2)$, and so on. We get a total of $5(4+3+2+1)$ rectangles with Southwest corner on the line $x = 1$. Continue.

Another way: We think the above approach is best, but there is a slicker way. Suppose that when we include the sides of the rectangle there are m East–West and n North–South lines. We produce a rectangle by *choosing* two East–West lines and two North–South lines to form its boundary. The East–West lines can be chosen in $\binom{m}{2}$ ways. For each such way, the North–South lines can be chosen in $\binom{n}{2}$ ways. Thus there are $\binom{m}{2}\binom{n}{2}$ rectangles. When $m = 5$ and $n = 7$ there are 210 rectangles.

Comment: Most students will not be familiar with the notation $\begin{pmatrix} n \\ r \end{pmatrix}$ for the number of ways of choosing r objects from n. But they will likely be familiar with the notation ${}_nC_r$.

Another way: We can imagine choosing a rectangle by (i) first choosing one of the 35 meeting points of our lines to serve as a corner of the rectangle—call this point P—and then (ii) choosing one of the 24 meeting points Q *not* on the same horizontal or vertical line as P to serve as a diagonally opposite corner of the rectangle. This procedure produces each rectangle $ABCD$ four times: $P = A$, $Q = C$; $P = C$, $Q = A$; $P = B$, $Q = D$; and $P = D$, $Q = B$. So the number of rectangles is $(35)(24)/4$. The same idea works generally.

7. Solve for x: $(x^2 - 6x)(x^2 - 6x + 6) = 16$.

Solution. A possibly fatal mistake is to multiply out: the given equation has a nice structure, and multiplying out would destroy that structure.

Let $u = x^2 - 6x$. Then our equation can be rewritten as

$$u(u+6) = 16 \qquad \text{or equivalently} \qquad u^2 + 6u - 16 = 0.$$

The last equation may be rewritten as $(u+8)(u-2) = 0$, and has the solutions $u = -8$ and $u = 2$.

So x is a solution of our original equation if and only if

$$x^2 - 6x = -8 \qquad \text{or} \qquad x^2 - 6x = 2.$$

The first equation can be rewritten as $x^2 - 6x + 8 = 0$; its roots are 2 and 4. The second equation can be rewritten as $x^2 - 6x - 2 = 0$. By the quadratic formula, its roots are given by

$$x = \frac{6 \pm \sqrt{44}}{2} = 3 \pm \sqrt{11}.$$

It would have been better to let $u = x^2 - 6x + 3$, halfway between $x^2 - 6x$ and $x^2 - 6x + 6$. This symmetrizing move simplifes things a bit. For then $x^2 - 6x = u - 3$ and $x^2 - 6x + 6 = u + 3$, so our equation becomes $(u-3)(u+3) = 16$, or equivalently $u^2 = 25$. Thus we have $u = \pm 5$, and we go on as before.

Another way: We can make the equation look nicer by completing the square in $x^2 - 6x$ and $x^2 - 6x + 6$. Note that

$$x^2 - 6x = x^2 - 6x + 9 - 9 = (x-3)^2 - 9.$$

Let $y = x - 3$. Then $x^2 - 6x = y^2 - 9$ and $x^2 - 6x + 6 = y^2 - 3$, so our equation becomes

$$(y^2 - 9)(y^2 - 3) = 16,$$

which expands to $y^4 - 12y^2 + 11 = 0$, a quadratic equation in y^2. By inspection, or by the quadratic formula, we find that $y^2 = 1$ or $y^2 = 11$ Thus $y = \pm 1$ or $y = \pm\sqrt{11}$. Add 3 to get x.

8. Al and Bob are having a two lap race in a 30 metre pool. Al swims the first lap freestyle at 2 metres per second. For the second lap he swims the backstroke at 1 m/s. Bob swims the butterfly at 1.5 m/s for the entire race. At what time(s) after the start will Al and Bob be side by side? (Al and Bob are very small—in fact they are points.)

Solution. Bob swims 60 metres at 1.5 metres per second, so he takes 60/1.5 seconds, that is, 40 seconds. Al takes 30/2 seconds for the first lap and 30/1 for the second, for a total of 45. Al loses.

But it is clear that Al finishes the first lap well ahead of Bob. So Bob and Al must meet once while Al is already on his second lap and Bob is still on his first. And they must meet again before the end of the race, since Bob will win.

"Algebra" gives us a fairly mechanical way of finding when they meet. But the algebra must be guided by a geometric view of the race.

Let's suppose Al and Bob first meet after t seconds. We have seen that Al is already on his second lap, so $t > 15$. Al has travelled a length of the pool, plus $(1)(t-15)$. And Bob has travelled a distance $1.5t$. We have

$$(1)(t-15) + 1.5t = 30,$$

and therefore $2.5t = 45$, so $t = 18$. Thus Al is already 3 seconds (metres) into his return trip; Al and Bob first meet 3 metres from the far end of the pool.

For the second time that they meet, again let it be after t seconds. Then Al has travelled a distance $30+(t-15)$. And Bob has travelled $1.5t$. These distance are equal, so $30 + (t-15) = 1.5t$. This simplifies to $15 = 0.5t$, giving $t = 30$. That's 15 seconds (metres) into Al's second lap, so Al and Bob meet for the second time at the midpoint of the pool.

Once we know that they first meet 3 metres from the far end, we can solve the problem without further algebra. For Bob needs to travel 2 seconds to complete his first lap. In that time Al has travelled 2 metres, so is 5 metres ahead. Bob now gains half a metre a second, so needs 10 seconds to catch Al. That puts him midway in the pool..

Another way: There is no need at all of formal algebra—some thinking is enough. Let's work backwards from the end. Bob took 40 seconds, Al took 45. So Bob won by 5 seconds, and at that time Al was 5 metres from finishing. Now run the movie backwards, so Al is swimming backwards at 1 metre per second, Bob at 1.5 metres per second, and Al has a 5 metre "lead." Bob gains half a metre a second, so in 10 seconds has caught up. That's 15 metres from the end, at the middle of the pool.

We can find the first meeting point in the same informal way. When Al finishes his first lap (15 seconds), Bob has travelled (15)(1.5), that is, 22.5, and therefore the two are 7.5 metres apart. Now they are travelling *towards* each other, and the gap between them is closing at $1 + 1.5$ metres per second. So it takes 3 seconds for them to meet after Al's turn.

Another way: In the picture, the horizontal axis represents time, and the vertical axis represents distance, as measured from the start of the race. The heavy line represents Al's space-time path, while the dashed heavy line represents Bob's. It is easy to draw these paths. Pick any convenient time scale, and any convenient distance scale. Al reaches the far end at time 15; his space-time position is then $(15, 30)$. Draw the line segment

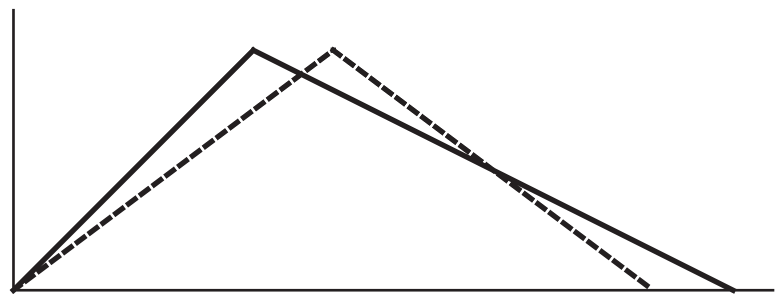

Figure 4: Al and Bob

joining $(0, 0)$ to $(15, 30)$. In another 30 seconds Al is finished, at position $(45, 0)$. Draw the line segment joining $(15, 30)$ to $(0, 45)$. We now have Al's space-time path. Bob's is drawn in a similar way.

The picture shows that the two space-time paths intersect twice. If we have drawn the picture carefully we can measure where in space-time they meet. Or else now that we have a purely geometric problem we can use other ideas from geometry to do the calculation.

9. The bisector of one of the acute angles of a right-angled triangle divides the opposite side into segments of length 7 and 25. Find the area of the triangle.

Solution. Please see Figure 5. Everything would be easy if we knew the length of AB. This we try to find. Draw the perpendicular from C to the

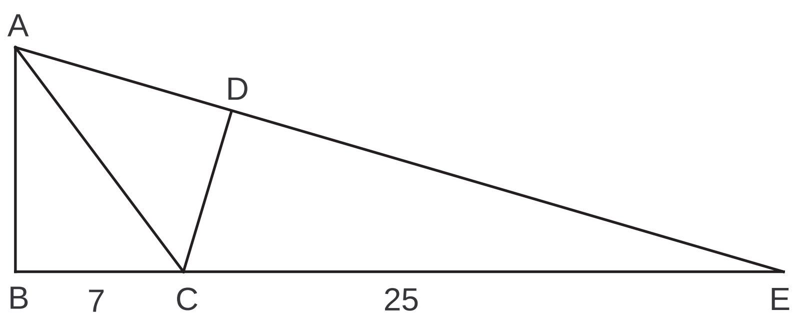

Figure 5: The Angle Bisector

line AE, meeting that line at D.

It is clear by symmetry that $\triangle ABC$ is congruent to $\triangle ADC$: the two angles at A match, so all angles match, and line AC is common.

It follows that CD has length 7. But then (Pythagorean Theorem) $DE^2 = 25^2 - 7^2$, and after a bit of calculation we find $DE = 24$. So the angle at E has tangent equal to $7/24$. But it also has tangent equal to $AB/(7+25)$. We conclude that $AB = 7(32/24)$. Finally, multiply AB by 32, divide by 2. If we simplify we get something like $448/3$.

Another way: We don't really need to know AB. Triangle CDE has area 84. But $\triangle ABE$ is a scaled-up version of $\triangle CDE$, with linear scaling factor $32/24$, that is, $4/3$. So areas scale by $(4/3)^2$, and after some simplification we get $448/3$.

10. Find all pairs (x, y) of positive integers such that $xy = 4x + 5y + 6$.

Solution. Rewrite the equation as $xy - 4x - 5y = 6$. We use an analogue of the familiar process of completing the square. Note that $xy - 4x - 5y$ is 'almost' $(x-5)(y-4)$. More precisely

$$xy - 4x - 5y = (x-5)(y-4) - 20,$$

so the original equation can be rewritten as

$$(x-5)(y-4) = 26.$$

We are looking for solutions (x, y) in positive integers. Let $u = x - 5$ and $v = y - 4$. Then u and v need to be integers with $uv = 26$. There aren't many possibilities.

Maybe $u = 1$, $v = 26$. That gives $x = 6$, $y = 30$.

Maybe $u = 26$, $v = 1$. That gives $x = 31$, $y = 5$.

Maybe $u = 2$, $v = 13$. That gives $x = 7$, $y = 17$.

Maybe $u = 13$, $v = 2$. That gives $x = 18$, $v = 6$.

One can also imagine u and v negative. But then in all cases one of x or y is negative.

Another way: Rewrite the equation as $xy - 5y = 4x + 6$. We can't have $x = 5$. For if $x = 5$ then the left side is 0 and the right side is not. So our equation is equivalent to

$$y = \frac{4x+6}{x-5}.$$

The nature of the right side becomes clearer if we divide $4x+6$ by $x-5$, using ordinary division of polynomials. The quotient is 4 and the remainder is 26, so our equation is equivalent to

$$y = 4 + \frac{26}{x-5}.$$

Since x and y are integers, it follows that $x - 5$ must be a factor of 26. Look first at the positive factors. Set $x - 5$ in turn equal to 1, 26, 2, or 13. Each choice gives a positive integer value for y.

We also need to examine negative factors of 26. The possibilities are $x - 5 = -13$, -26, -1, and -2. The first two give negative values of x and the last two give negative values of y.

Another way: We can do a cruder investigation based on size estimates. The intuition behind it is that if x and y are both big, then xy is *bigger* than $4x+5y+6$. So if our equation holds, the smaller of x and y can't be large. Then we do a crude search through small x and y. Bounds followed by a search are a useful tool, particularly if we can make a computer do the searching.

Suppose first that $x \le y$. Then $4x+5y+6 \le 9y+6$. So if $xy = 4x+5y+6$ then $xy \le 9y+6$. This forces $x < 10$. Similarly, if $y \le x$ then $y < 10$. We conclude that $\min(x, y) < 10$.

The above observations limit the hunt, and we can limit it further. From $xy = 4x + 5y + 6$ we conclude that $x > 5$ (else the left side would be too small) and $y > 4$ (same reason). Thus we need only explore the possibilities $6 \le x \le 9$, (with no conditions yet on y) and $5 \le y \le 9$ (with no conditions yet on x.)

Put $x = 6$. Our equation becomes $6y = 5y + 30$, so $y = 30$. Next put $x = 7$. Our equation becomes $7y = 34 + 5y$, so $y = 17$. Put $x = 8$. We get $8y = 38 + 5y$, which has no integer solution. A similar problem arises with $x = 9$. Now deal in the same way with $y = 5, 6, \dots, 9$. We quickly find the other two solutions.

11. A cat owns 4 identical socks and 4 identical boots. In how many orders can it put on socks and boots in the morning? There is no such thing as a left cat boot or a right cat boot. And a sock must go on a paw before—but not necessarily immediately before—a boot is put on that paw.

Solution. We can make a reasonably well-organized list. But some shortcuts are necessary, since it will turn out that the number of ways is 2520. Here is one way of doing the listing.

There are 8 actions that the cat will take, one after the other, say at times 1, 2, 3, 4, 5, 6, 7, and 8. At time 1, the cat must put a sock on one of its paws. There are 4 choices for *which* paw is the first to get socked. And for every one of these choices, there remain 7 times for *when* that paw will get booted, for a total of $4 \cdot 7$ ways.

For each of the $4 \cdot 7$ possibilities described above, there are 3 choices for which is the second paw to get socked. And for each such choice, there are 5 possible times *when* that second paw get booted. So up to now we have $(4 \cdot 7)(3 \cdot 5)$ choices. For each of these choices, there are 2 choices for which is the third paw to get socked. And for each such choice, there are 3 times *when* that third paw gets booted. Now everything is determined, so the number of ways for the cat to be well-shod is $(4 \cdot 7)(3 \cdot 5)(2 \cdot 3)$, that is, 2520.

Another way: Label the cat's feet 1, 2, 3, and 4. The actions of the cat can be captured as an 8-digit number made up of two occurrences each of the digits 1, 2, 3, and 4. So for example the number 21234413 means that the cat first put a sock on paw 2, then a sock on paw 1, then a boot on paw 2, then a sock on 3, then a sock on 4, then a boot on 4, a boot on 1, and a boot on 3. In each case, the first occurrence of a digit refers to putting on the sock, and the second refers to putting on the boot.

We must choose two places for the 1's to go. This can be done in $\binom{8}{2}$ ways. For each such choice, there are $\binom{6}{2}$ ways of placing the 2's. So the first two tasks can be done in $\binom{8}{2}\binom{6}{2}$ ways. And for each of these ways the 3's can be placed in $\binom{4}{2}$ ways. Once the 3's have been placed, there is only one way of placing the 4's. To make things look nice let's call this $\binom{2}{2}$. So the total number of ways of putting on socks and boots is

$$\binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{2}{2}.$$

(Students will be more familiar with the notation ${}_nC_r$ for the binomial coefficients.) Compute. We get $(8 \cdot 7)(6 \cdot 5)(4 \cdot 3)(2 \cdot 1)/2^4$.

Another way: This is not *really* different. Represent the actions of the cat by a sequence of the 8 different symbols s_1, s_2, s_3, s_4, b_1, b_2, b_3, b_4. For example $s_2s_1b_2s_3s_4b_4b_1b_3$ means that a sock was put on paw 2, then a sock on paw 2, then a boot on paw 2, and so on. We must count the number of sequences in which for any i, s_i comes before b_i.

There is a total of 8! sequences. By symmetry, in half of these s_1 comes before b_1, and in half it comes after. So there are $8!/2$ sequences in which s_1 comes before b_1. Among these sequences, half have s_2 coming before b_2, and half after. So there are $8!/2^2$ sequences in which s_1 comes before b_1 *and* s_2 comes before b_2. And in half these sequences, s_3 comes before b_3. Go on in this way. We conclude that there are $8!/2^4$ permissible patterns.

The arguments used for ordinary cats work equally well for n-footed cats. The number of ways turns out to be $(2n)!/2^n$.

12. (i) Find integers a and b such that $0 < a - b\sqrt{2} < 0.6$. (ii) Find integers a and b such that $0 < a - b\sqrt{2} < 0.36$. (iii) Find integers a and b such that $0 < a - b\sqrt{2} < 0.01$.

Solution. (i) We can find $\sqrt{2}$ approximately without a calculator by an informal approximation procedure: $14^2 = 196$ and $15^2 = 225$, so $\sqrt{2}$ is between 1.4 and 1.5, and (probably) closer to 1.4. It should not be hard to see that $0 < 2 - \sqrt{2} < 0.6$. So we can choose $a = 2$ and $b = 1$. There are infinitely many other answers.

(ii) One can in fact easily beat 0.36. For example we might note that $2\sqrt{2}$ lies between 2.8 and 3, so if $a = 3$ and $b = 2$ we have $0 < a - b\sqrt{2} < 0.2$.

The 0.36 in the question is meant to encourage squaring. We have

$$0 < 2 - \sqrt{2} < 0.6 \qquad \text{and therefore} \qquad 0 < (2 - \sqrt{2})^2 < 0.36.$$

Expand $(2 - \sqrt{2})^2$. We get $6 - 4\sqrt{2}$. So we can pick $a = 6$ and $b = 4$. But immediately we notice that we can do better and more cheaply. Divide by 2 and we get $0 < 3 - 2\sqrt{2} < 0.18$.

(iii) We can for example square both sides of the inequality

$$0 < 3 - 2\sqrt{2} < 0.18.$$

(With a calculator, we could note that in fact $3 - 2\sqrt{2} < 0.172$, but this sort of improvement is not of great value.)

Since $(3 - 2\sqrt{2})^2 = 17 - 12\sqrt{2}$, we conclude that $17 - 12\sqrt{2} < .04$, in fact $17 - 12\sqrt{2} < .03$. Square again. We get that $577 - 408\sqrt{2}$ is sufficiently small. In fact (calculator) it is about 9×10^{-4}.

13. Let P be the point with coordinates $(4, 6)$ and Q the point with coordinates $(0, 3)$. Find the coordinates of the point(s) R on the x-axis such that $\triangle PQR$ has area 20.

Solution. A picture like Figure 6 is more or less essential. Think of our

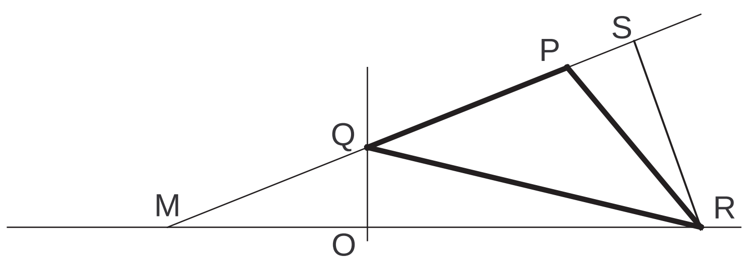

Figure 6: Making a Triangle with Area 20

triangle as having base PQ. A quick computation shows that

$$PQ = \sqrt{(4 - 0)^2 + (6 - 3)^2} = 5.$$

Pleasant. Since the area of a triangle is half of base times height, we want our triangle to have height 8.

We are therefore looking for a point on the x-axis which is at distance 8 from the line through P and Q. A look at the picture shows that for any $d > 0$, there are *two* points on the x-axis which are at distance d from the line PQ. One of them is a certain distance to the *right* of the point we have marked M, and the other is the same distance to the *left* of M.

This is clear by symmetry: if we continue the line PQ to the left of M, we get a mirror image of what happens to the right of M.

Imagine that R is the point to the right of M that does the job, and draw a perpendicular from R to PQ, meeting PQ say at S. Then $RS = 8$.

The line PQ has slope $3/4$. Since $OQ = 3$, we have $OM = 4$, and therefore $MQ = 5$.

But triangles OQM and RSM are similar, $RS/MR = OQ/MQ = 3/5$. But $RS = 8$, so $MR = 40/3$. It follows that $OQ = 40/3 - 4 = 28/3$, and this gives the x-coordinate of R as drawn. But there is another point on the other side of M that does the job, say R'. Then $MR' = MR$, so the x-coordinate of R' is $-4 - 40/3$, or $-52/3$.