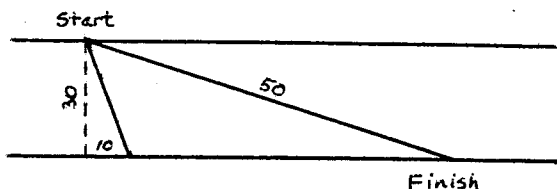


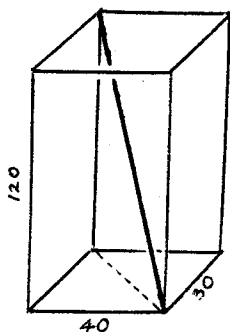
# UBC Grade 8–9 Workshop Solutions 2003

1. Single-variable algebra gives an efficient solution. Let  $x$  be the percentage score on the last exam. If  $x$  turns out to be greater than 100, then an  $A$  is impossible. To find  $x$ , require the average score to be at least 86 (it may be useful to review the definition of the average):  $(80 + 78 + 97 + 76 + x)/5 \geq 86$ , which gives  $x \geq 99$ . So Laura can get an  $A$  in math, if she scores at least 99% on the last test.
2. The phrase “5% more Real Fuzz than before” could be interpreted in different ways. It might mean that the new product is  $75\% + 5\% = 80\%$  Real Fuzz. Or it may mean that the amount of Real Fuzz in the old 500 mL can has been increased by 5%. If this is the case, then the amount of Real Fuzz in the old can is 75% of 500 mL  $= .75 \times 500 = 375$  mL of Real Fuzz. Then 5% of 375 mL is  $.05 \times 375 = 18.75$  mL. Adding this amount to 375 mL gives  $375 + 18.75 = 393.75$  mL of Real Fuzz. If this amount is in the 700 mL bottle of cola, the percentage of Real Fuzz in the new product is only  $393.75/700 \times 100\% = 56.25\%$ . Another interpretation is that the new product has  $75\% + 5\% = 80\%$  of 500 mL, or 400 mL of Real Fuzz, which as a percentage of 700 mL is  $400/700 \times 100\% = 57.14\%$ . If either of the last two interpretations is what the company uses, the new product has proportionately less Real Fuzz than the old product.
3. A picture, or even a supply of real blocks, would be useful to help students understand the solution. Since  $12/3 = 4$ , we can fit 4 of the solid cubes along one edge of the empty box. Therefore,  $4 \times 4 \times 4 = 4^3 = 64$  cubes could fit inside the box. If the edge of the empty box is twice as long, we could fit  $2 \times 4$  solid cubes along one edge of the box, and  $(2 \times 4) \times (2 \times 4) \times (2 \times 4) = 2^3 \times 4^3$  solid cubes could fit inside the box. Similarly, if the edge of the box is 3 times as long,  $3^3 \times 4^3$  solid cubes could fit inside the box, and if the edge of the box is  $n$  times as long, then  $n^3 \times 4^3$  solid cubes could fit inside the box.
4. Single-variable algebra can be used. Let  $x$  be the amount of pumpkins the store originally had. Then  $100\% - 70\% = 30\%$  of  $x$  pumpkins, or  $0.3x$  pumpkins are unsold. We are told that 80% of  $0.3x$  is 240 pumpkins, or  $0.8(0.3x) = 240$ . Solving for  $x$  gives  $x = 1000$  pumpkins.
5. We want the total cost, which is  $\$3.00(\text{no. of pineapples}) + \$1.55(\text{no. of mangos}) + \$2.75(\text{no. of watermelons})$ , to be at most \$111.00. Since the numbers of pineapples, mangos and watermelons are the same, they can all be represented by the same variable  $x$ , so we require  $3x + 1.55x + 2.75x \leq 111$ . Solving for  $x$  gives  $x \leq 15.2$ , so Mike can buy  $x = 15$  of each fruit.
6. Draw a diagram. Team one travels 50 m, along the hypotenuse of a right triangle which has length 50 m. It is probably clear to most students that Team 2 travels farther. To find how far, we need to find the distance travelled across the river, and also the distance run along the river bank. The distance Team 2 travels across the river is along the hypotenuse of a right triangle whose other two sides have lengths 30 m and 10 m. By the Pythagorean Theorem, the hypotenuse has length  $\sqrt{30^2 + 10^2} = \sqrt{1000} = 10\sqrt{10} \approx 31.62$  m. To find how far Team two runs, we need to find how far downstream the finish point is. Looking at the first, larger right triangle and using the Pythagorean Theorem again, this distance is  $\sqrt{50^2 - 30^2} = 40$  m, so Team two has to

run  $40 - 10 = 30$  m. The total distance Team two travels is  $30 + 10\sqrt{10}$  m, which is  $30 + 10\sqrt{10} - 50 = 10\sqrt{10} - 20 \approx 11.62$  m farther than Team one travels.

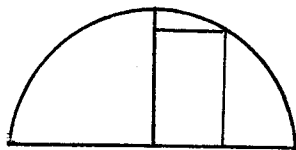


7. Draw a diagram. We need to find the distance between opposite corners of the box. This can be found in two steps, using the Pythagorean Theorem in both steps. The distance between corners of the rectangular base of the box is  $\sqrt{30^2 + 40^2} = 50$  cm, and this can be thought of as the base of a right triangle whose hypotenuse is the thin straight rod. The other side of this latter right triangle is 120 cm long, so the rod is  $\sqrt{50^2 + 120^2} = 130$  cm long. Alternatively (noting that  $50^2 = 30^2 + 40^2$ ), the rod length can be calculated directly as  $\sqrt{30^2 + 40^2 + 120^2}$ , and this example could be generalized to derive the distance between opposite corners of any three-dimensional box, as  $\sqrt{x^2 + y^2 + z^2}$ , where  $x$ ,  $y$  and  $z$  are the lengths of the sides of the box.



8. This question is about rates. A diagram or a table, or both, can be used to help explain the solution. The key idea is to find the distance Ed rides in half an hour, since for Ed the “race” distance is 50 km minus the distance he rides in a half hour. Travelling at 25 km per hour for  $\frac{1}{2}$  hour covers a distance of  $25 \times \frac{1}{2} = 12.5$  km, so the race distance for Ed is  $50 - 12.5 = 37.5$  km. He covers this distance in  $37.5/25 = 1.5$  hours. On the other hand, Fred must cover a distance of 50 km travelling at 32 km per hour. This takes Fred  $50/32 = 1.5625$  hours, so Ed wins the race.
9. Probably the most straightforward way to find the area of the quadrilateral  $ACDE$  is to subtract the areas of the triangles  $ABC$  and  $AEF$  from the area of the square  $ABDF$ . One side of the square has length  $56/4 = 14$  cm, so the area of the square is  $14^2 = 196$  cm<sup>2</sup>. The area of triangle  $ABC$  is  $\frac{1}{2}(14)(8) = 56$  cm<sup>2</sup>, and the area of triangle  $AEF$  is  $\frac{1}{2}(14)(6) = 42$  cm<sup>2</sup>. Therefore the area of quadrilateral  $ACDE$  is  $196 - 56 - 42 = 98$  cm<sup>2</sup>.
10. Draw a picture. To see if the truck will fit in the tunnel, one side of it should be as close to the middle of the tunnel as possible. So a rectangle 7 m high and 4 m wide should fit into a quarter of a circular disk that represents half of the tunnel. The distance from the middle of the tunnel to the edge is the radius of the disk, so the

radius is  $16/2 = 8$  m. Referring to the diagram, we can see that for the truck to fit the diagonal measurement of the rectangle can be at most 8 m. By the Pythagorean Theorem, the diagonal of the rectangle has length  $\sqrt{7^2 + 4^2} = \sqrt{65}$  m, which is larger than 8 m (evaluating  $\sqrt{65} \approx 8.06$  may be helpful, but not necessary). So the truck won't fit in the tunnel.



11. The number of different combinations can be counted, if the counting is done systematically. A tree diagram or similar visual representation could be used. Since the red and green bulbs are always adjacent, treat them as one object initially. Then we have four objects to be placed in four slots. In the first slot there are four possible choices. After making the first choice, then in the second slot there are three possible choices. After making the first and second choices for the third slot there are two possible choices, and in the last slot only one choice. All the choices can be counted, and there are  $4 \times 3 \times 2 \times 1 = 4! = 24$  possibilities treating the red and green bulbs as one object. But the red and green bulbs can be in either order so in all there are  $24 \times 2 = 48$  possibilities.
12. We start by keeping track of how many new, day old, and deceased glibbles there are on each day, and then discover a pattern. A chart could be made showing the day, the number of new glibbles, the number of day old glibbles and the number of deceased glibbles. On Day 1 there is one new glibble. On Day 2 the original glibble has grown one day old and has given birth, so there is one new glibble and one day old glibble. On Day 3 the glibbles alive on the previous day have all given birth and grown one day older, so there are two new glibbles, one day old glibble and one deceased glibble. On each day the number of new glibbles is the sum of the previous day's numbers of new glibbles and day old glibbles, while the number of day old glibbles is the previous day's number of new glibbles. We don't really need to keep track of deceased glibbles. The number of new glibbles on each day follows the Fibonacci sequence, and so does the number of day old glibbles, shifted by one day. On Day 15, there are 610 new glibbles and 377 day old glibbles, for a total of 987 live glibbles. Recognizing the pattern of the Fibonacci sequence reduces the amount of work required.
13. First make sure students understand the problem. The number keys for 9 and 1 must be working, so either the 2 key or the 3 key is broken. No matter which of these two keys is broken, the phone number would not begin with 911 (otherwise anyone trying to dial the number would always connect with the Emergency Number). A tree diagram can be used to help enumerate all possibilities. Suppose the 2 key is broken. The 9 must come before two 1's with zero, one or two 2's mixed in, but a 3 cannot come before a 1, so the first digit is either a 9 or a 2. Suppose the first digit is a 9. Then the next digit is a 1 or a 2. Reasoning in this way (see the diagram), the nine

Figure 1 is a tree diagram illustrating the decomposition of the representation 9 into irreducible representations of the symmetric group  $S_4$ . The root node is 9, which branches into 1-2 and 2. The 1-2 node branches into 1 and 2-1-3-3. The 2 node branches into 1 and 2-1-1-3-3. The 1 node branches into 2-3-3 and 3-2-3. The 2-1-3-3 node branches into 2-3-3 and 3-2-3. The 2-1-1-3-3 node branches into 2-3-3 and 3-2-3.

2-9-1-1-3-3

2

9

1

1

2-3-3

3

2-3

3-2

2-1-3-3

2-1-1-3-3

15. The solution is much easier if we imagine we are sitting in the limousine and observing

the car go by (*i.e.* transform to the reference frame of the limousine). Relative to the limousine, the car is travelling at  $30 + 24 = 54$  m/s. Draw a diagram. If the limousine is  $x$  m long, then the car moves  $x + 3$  m in  $\frac{1}{6}$  s. Since the rate the car passes by is 54 m/s, we have  $x + 3 = (54) \left(\frac{1}{6}\right)$ , and solving for  $x$  gives the length of the limousine  $x = 6$  m

16. We can try a few powers of 3 and see if a pattern can be found.  $3^1 = 3$ ,  $3^2 = 9$ ,  $3^3 = 27$ ,  $3^4 = 81$ ,  $3^5 = 243$  and so on. After writing down higher and higher powers it may be noticed that the last digits seem to repeat in a cycle of four: 3, 9, 7, 1, 3, 9, 7, 1, ... If this is indeed the case then since 2000 is divisible by four, the last digit of  $3^{2000}$  must be a 1, so the last digit of  $3^{2001}$  is a 3, the last digit of  $3^{2002}$  is a 9, and the last digit of  $3^{2003}$  is a 7. A way to *prove* that the last digit of  $3^{2003}$  is a 7 is to write  $3^{2003} = 3^{2000+3} = 3^{2000}3^3 = 3^{4 \cdot 500}3^3 = (3^4)^{500}3^3 = 81^{500}3^3$ . Since the last digit of 81 is a 1, the last digit of any positive integer power of 81 is also a 1. The last digit of  $3^3$  is a 7, so the last digit of  $81^{500}3^3$  is  $1 \cdot 7 = 7$ .
17. This problem can be solved by carefully listing all outcomes that result in Keenan winning an amount greater than or equal to \$2000, and calculating the probability of each of these outcomes using laws of probability. The sum of the probabilities of these outcomes is the probability that Keenan will win at least \$2000. For the first choice, Keenan can pick a card numbered 1, 2 or 3.

If the card is numbered 1, then Keenan can win at most \$1000 so we don't have to calculate the probabilities of any outcomes that begin with choosing card 1.

If Keenan picks the card numbered 2, then to win \$2000 he must choose a green chip on both of the draws from the bag. The probability of choosing any one of the three cards is  $\frac{1}{3}$ . The probability of choosing a green chip on the first draw is  $\frac{3}{5}$ . There are then four chips left, two of which are green, so the probability of choosing another green chip on the second draw is  $\frac{2}{4} = \frac{1}{2}$ . Thus the outcome

2-green-green has probability  $\frac{1}{3} \cdot \frac{3}{5} \cdot \frac{1}{2} = \frac{1}{10}$ .

If he picks the card numbered 3, then in the three draws from the bag he can pick two green chips and a red one (winning \$2000), or three green chips (winning \$3000). The outcomes can be labelled

3-green-green-red, with probability  $\frac{1}{3} \cdot \frac{3}{5} \cdot \frac{2}{4} \cdot \frac{2}{3} = \frac{1}{15}$ ,

3-green-red-green, with probability  $\frac{1}{3} \cdot \frac{3}{5} \cdot \frac{2}{4} \cdot \frac{2}{3} = \frac{1}{15}$ ,

3-red-green-green, with probability  $\frac{1}{3} \cdot \frac{2}{5} \cdot \frac{3}{4} \cdot \frac{2}{3} = \frac{1}{15}$ ,

3-green-green-green, with probability  $\frac{1}{3} \cdot \frac{3}{5} \cdot \frac{2}{4} \cdot \frac{1}{3} = \frac{1}{30}$ .

So the probability of winning at least \$2000 is  $\frac{1}{10} + \frac{1}{15} + \frac{1}{15} + \frac{1}{15} + \frac{1}{30} = \frac{10}{30} = \frac{1}{3}$  (or  $33\frac{1}{3}\%$ ).

18. For motivation we could list several numbers to get some ideas. Of the first ten fractions  $\frac{1}{10000}$ ,  $\frac{2}{10000}$ ,  $\frac{3}{10000}$ ,  $\frac{4}{10000}$ ,  $\frac{5}{10000}$ ,  $\frac{6}{10000}$ ,  $\frac{7}{10000}$ ,  $\frac{8}{10000}$ ,  $\frac{9}{10000}$ ,  $\frac{10}{10000}$  we see that if the numerator is an even number 2, 4, 6, 8, 10 then the corresponding fraction is not reduced to lowest terms because both the numerator and the denominator have a common factor 2. Similarly if the numerator is 5 then  $\frac{5}{10000}$  is not reduced to lowest terms because the numerator and the denominator have a common factor 5. Notice that  $10000 = 10^4 = (2 \cdot 5)^4 = 2^4 5^4$  so the only prime factors of 10000 are 2 and 5, thus any factor of 10000 is divisible by 2 or 5 (or both, which is implicit when we use

the word 'or'). Therefore the *only* fractions that are not reduced to lowest terms have numerators that are divisible by 2 or 5. These are the numerators that end in 2, 4, 5, 6, 8 or 0: six out each group of ten consecutive numerators. Thus four out each group of ten consecutive fractions in the set are reduced to lowest terms, for a total of  $4 \cdot 1000 = 4000$  numbers that are fractions reduced to lowest terms.