UBC Math 11/12 Workshop Solutions 2003

1. This is straightforward to solve, using algebra. The total cost of the 50 kg mixture is $(15.2)(50) = 760$ dollars. Let x be the amount (in kg) of coffee worth \$14/kg. Then the amount of coffee worth \$16/kg in the mixture is $50 - x$ kg. The total cost of the mixture is $14x + 16(50 - x) = 760$ dollars, and solving gives $x = 20$ kg. Thus the grocer needs 20 kg of the coffee worth \$14/kg and 30 kg of the coffee worth \$16/kg.

2. This question is about rates. A diagram or a table, or both, can be used to help explain the solution. The key idea is to find the distance Ed rides in half an hour, since for Ed the "race" distance is 50 km minus the distance he rides in a half hour. Travelling at 25 km per hour for $\frac{1}{2}$ hour covers a distance of $25 \times \frac{1}{2} = 12.5$ km, so the race distance for Ed is $50 - 12.5 = 37.5$ km. He covers this distance in $37.5/25 = 1.5$ hours. On the other hand, Fred must cover a distance of 50 km travelling at 32 km per hour. This takes Fred $50/32 = 1.5625$ hours, so Ed wins the race.

3. The number of different combinations can be counted, if the counting is done systematically. A tree diagram or similar visual representation could be used. Since the red and green bulbs are always adjacent, treat them as one object initially. Then we have four objects to be placed in four slots. In the first slot there are four possible choices. After making the first choice, then in the second slot there are three possible choices. After making the first and second choices for the third slot there are two possible choices, and in the last slot only one choice. All the choices can be counted, and there are $4 \times 3 \times 2 \times 1 = 4! = 24$ possibilities treating the red and green bulbs as one object. But the red and green bulbs can be in either order so in all there are $24 \times 2 = 48$ possibilities. Students familiar with these ideas should be able to solve the problem quickly if they initially treat the red and green bulbs together as one object.

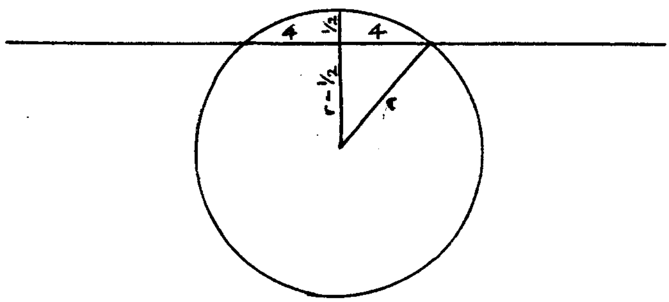
4. We can start by keeping track of how many new (age less than one day), day old (age at least one day and less than two days), and deceased glibbles there are on each day, and then discover a pattern. It is helpful to make a chart showing the day, the number of new glibbles, the number of day old glibbles and the number of deceased glibbles. On Day 1 there is one new glibble. On Day 2 the original glibble has become a day old glibble and has given birth, so there is one new glibble and one day old glibble. On Day 3 the glibbles alive on the previous day have all given birth and grown one day older, so there are two new glibbles, one day old glibble and one deceased glibble. On each day the number of new glibbles is the sum of the previous day's numbers of new glibbles and day old glibbles, while the number of day old glibbles is the previous day's number of new glibbles. We don't really need to keep track of deceased glibbles. The number of new glibbles on each day follows the Fibonacci sequence, and so does the number of day old glibbles, shifted by one day. On Day 15, there are 610 new glibbles and 377 day old glibbles, for a total of 987 live glibbles. Recognizing the pattern of the Fibonacci sequence reduces the amount of work required.

5. The solution is straightforward if we imagine we are sitting in the limousine and observing the car go by (*i.e.* transform to the reference frame of the limousine). Relative to the limousine, the car is travelling at $30 + 24 = 54$ m/s. Draw a diagram. If the limousine is x m long, then the car moves $x + 3$ m in $\frac{1}{6}$ s. Since the rate the car

passes by is 54 m/s, we have $x+3=(54)\left(\frac{1}{6}\right)$, and solving for x gives the length of the limousine $x=6$ m

6. Assuming that the width of the staircase is much smaller than the diameter of the building, the staircase can be represented by a line on the outside wall of the building. Imagine cutting the outside wall vertically between the top and bottom of the staircase and "unwrapping" the wall to form a rectangle. If the steps on the staircase are all the same size, then it can now be represented by a straight line on the rectangle joining opposite corners, and its length is given by the Pythogarean theorem. The height of the rectangle is the height of the building, 100 m, and the width of the rectangle is the circumference of the building, 60π m. Therefore the length of the staircase is $\sqrt{100^2+(60\pi)^2}\approx 213.38$ m.

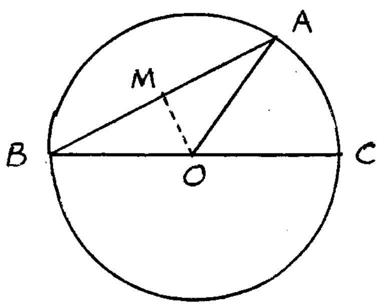
7. To find the volume of a sphere, we need to know its radius. To find the radius, draw a cross section of the sphere in the sand, assuming the ocean floor is flat. By the



Pythogarean Theorem, $r^2=\left(r-\frac{1}{2}\right)^2+4^2$, and solving gives $r=16\frac{1}{4}=\frac{65}{4}$. Therefore the volume of the spaceship is $\frac{4}{3}\pi r^3=\frac{4}{3}\pi\left(\frac{65}{4}\right)^3\approx 17974.16\ \text{m}^3$.

8. The total volume of water is equal to the sum of the volumes of columns of water sitting above each of the three sections. The volume of each column of water is the product of the depth and the area of the section as seen from above. The area of the deep (3 m) section is half the area of the circular disk of radius $36/2=18$ m, so the area of the deep section is $\frac{1}{2}\pi 18^2\approx 508.94\ \text{m}^2$.

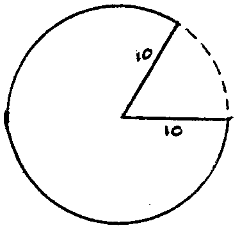
To find the area of the medium section ABC, draw a line from the point A to the centre of the circle O. Then the triangle AOB is isosceles (be prepared to explain

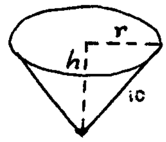


why), so angle BAO is 30°, angle AOB is 120°, and angle AOC is 60°. The area of the medium section ABC is the area of the sector AOC plus the area of the isosceles triangle AOB. The area of the sector AOC is $\pi 18^2\left(\frac{60}{360}\right) \approx 169.65$ m^2. To find the area of the isosceles triangle AOB, locate the midpoint M of AB. Then the height of the triangle is OM, which by trigonometry is $18\sin 30° = 9$ m, and the base AB of the triangle has length $2 \cdot 18\cos 30° = 18\sqrt{3} \approx 31.177$ m, so the area of the triangle is $\frac{1}{2}(9)(18\sqrt{3}) = 81\sqrt{3} \approx 140.30$ m^2. Therefore the area of the medium (2 m deep) section is $\frac{1}{6}\pi 18^2 + 81\sqrt{3} \approx 309.94$ m^2.

The area of the shallow (1 m deep) section is $\frac{1}{2}\pi 18^2 - \frac{1}{6}\pi 18^2 - 81\sqrt{3} \approx 199.00$ m^2, and finally, the volume of water needed to fill the pool is $3\left(\frac{1}{2}\pi 18^2\right) + 2\left(\frac{1}{6}\pi 18^2 + 81\sqrt{3}\right) + 1\left(\frac{1}{3}\pi 18^2 - 81\sqrt{3}\right) = \frac{13}{6}\pi 18^2 + 81\sqrt{3} \approx 2345.69$ m^3. Keeping exact expressions throughout the calculation reduces accumulated roundoff errors.

9. Try giving a clue first: find the circumference of the cone. If the circumference of the cone is known, then the radius r and height h can be determined, and the volume of the cone is $V = \frac{1}{3}\pi r^2 h$. Since a sector of angle 72° is removed, what remains is a sector of angle $360° - 72° = 288°$. The circular edge of this remaining sector has length $\pi(20)(288/360) = 16\pi$ cm, and when the straight edges are joined this length becomes the circumference of the cone. Therefore the radius of the cone is $r = 8$ cm, and by the Pythogarean Theorem, the height is $h = \sqrt{10^2 - 8^2} = 6$ cm. The volume of the cone is then $\frac{1}{3}\pi(8^2)(6) = 128\pi \approx 402.12$ cm^3.



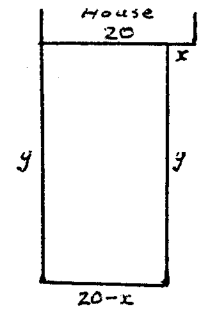


10. We can try a few powers of 3 and see if a pattern can be found. $3^1 = 3$, $3^2 = 9$, $3^3 = 27$, $3^4 = 81$, $3^5 = 243$ and so on. After writing down higher and higher powers it may be noticed that the last digits seem to repeat in a cycle of four: $3, 9, 7, 1, 3, 9, 7, 1, \ldots$. If this is indeed the case then since 2000 is divisible by four, the last digit of 3^{2000} must be a 1, so the last digit of 3^{2001} is a 3, the last digit of 3^{2002} is a 9, and the last digit of 3^{2003} is a 7. A way to *prove* that the last digit of 3^{2003} is a 7 is to write $3^{2003} = 3^{2000+3} = 3^{2000}3^3 = 3^{4\cdot 500}3^3 = (3^4)^{500}3^3 = 81^{500}3^3$. Since the last digit of 81 is a 1, the last digit of any positive integer power of 81 is also a 1. The last digit of 3^3 is a 7, so the last digit of $81^{500}3^3$ is $1 \cdot 7 = 7$.

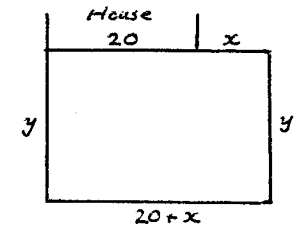
11. For motivation we could list several numbers to get some ideas. Of the first ten fractions $\frac{1}{10000}, \frac{2}{10000}, \frac{3}{10000}, \frac{4}{10000}, \frac{5}{10000}, \frac{6}{10000}, \frac{7}{10000}, \frac{8}{10000}, \frac{9}{10000}, \frac{10}{10000}$ we see that if the numerator is an even number 2, 4, 6, 8, 10 then the corresponding fraction is not reduced to lowest terms because both the numerator and the denominator have a common factor 2. Similarly if the numerator is 5 then $\frac{5}{10000}$ is not reduced to lowest terms because the numerator and the denominator have a common factor 5. Notice that $10000 = 10^4 = (2\cdot 5)^4 = 2^4 5^4$ so the only prime factors of 10000 are 2 and 5, thus any factor of 10000 is divisible by 2 or 5 (or both, which is implicit when we use the word 'or'). Therefore the *only* fractions that are not reduced to lowest terms have numerators that are divisible by 2 or 5. These are the numerators that end in 2, 4,

5, 6, 8 or 0: six out each group of ten consecutive numerators. Thus four out each group of ten consecutive fractions in the set are reduced to lowest terms, for a total of $4 \cdot 1000 = 4000$ numbers that are fractions reduced to lowest terms.

12. Since we can't be sure whether all or part of the side of Katie's house should be used, we must consider two cases. In both cases we find a quadratic polynomial to maximize.



Case 1



Case 2

In Case 1 we assume part of the side of the house is used. Let x denote the distance along the unused part of the house, $0 \leq x \leq 20$, so the dimensions of the rectangular enclosure are $(20 - x) \times y$. Since the fencing is 80 m long we have $2y + (20 - x) = 80$, thus $y = 30 + \frac{1}{2}x$, and the area of the enclosure is $A_1 = (20 - x)y = (20 - x)(30 + x/2)$. To find the maximum (without using calculus!), we can expand A_1 and complete the square to get $A_1 = 800 - (x + 20)^2/2$. This is maximized when $(x + 20)^2$ is as small as possible, which is when $x = 0$ (use all of the side of the house for one side of the enclosure), and in this case we get the maximum possible area in Case 1 as $A_1 = 600$.

In Case 2 all of the side of the house is used and the side of the enclosure may go beyond the house. Let x denote the distance the enclosure sticks out past the house, $0 \leq x \leq 30$ ($x \leq 30$ because there is only 80 m of fencing available), so the dimensions of the rectangular enclosure are $(20 + x) \times y$. Again using the fact that the fencing is 80 m long, we get this time $2y + x + (20 + x) = 80$, or $y = 30 - x$, and therefore the area of the enclosure is $A_2 = (20 + x)y = (20 + x)(30 - x)$. Expanding and completing the square gives $A_2 = 625 - (x - 5)^2$. This is maximized when $x = 5$ (any other value of x would reduce A_2 from 625), so the maximum possible area in Case 2 is $A_2 = 625$.

Since the maximum in Case 2 is larger than the maximum in Case 1, Case 2 gives the maximum area for both cases, which is 625 m^2. Some students may reason that the maximum area corresponds to a square enclosure, and this leads immediately to 625 m^2, but this only applies to Case 2, and they should be asked to explain why Case 1 could not give a larger area (perhaps they did not consider Case 1).

13. This problem can be solved using algebra. We get four equations in four unknowns, but the solution does not require advanced techniques. Let x be the regular price of a pair of pants, let y be the regular price of a shirt, and let z be the regular price of a sweater. The third sentence of the question can be expressed as $2x + 2y + 2z = 230$, or equivalently,

$$x + y + z = 115. \qquad (1)$$

During the sale, two pairs of pants sell for $x + 0.8x = 1.8x$, so Irving's purchases give a total cost of

$$1.8x + y + z = 155. \tag{2}$$

Similarly (three shirts on sale cost $y + 0.8y + 0.7y = 2.5y$), Jerome's purchases give

$$x + 2.5y + 1.8z = 184.5, \tag{3}$$

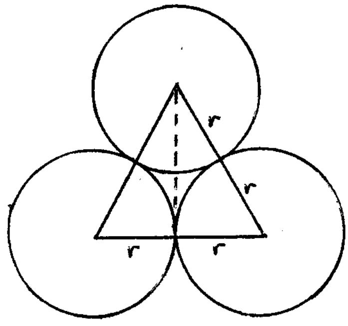
and Keenan's purchase give

$$2.5x + 1.8y + kz = 270, \tag{4}$$

where k is also an unknown, related to the number of sweaters that Keenan buys. Subtracting (1) from (2) gives $0.8x = 40$, so the regular price of a pair of pants is $x = 50$ dollars. Now substituting $x = 50$ into (2) and (3) gives two linear equations $y + z = 65$ and $2.5y + 1.8z = 134.5$ in the two unknowns y and z, which can be solved (various ways are known to Math 11 students) to get $y = 25$ and $z = 40$. So the regular price of a shirt is \$25 and the regular price of a sweater is \$40. Finally, substituting $x = 50$, $y = 25$ and $z = 40$ into (4) and solving gives $k = 2.5$, which we recognize as $2.5 = 1 + 0.8 + 0.7$, so Keenan bought 3 sweaters on sale.

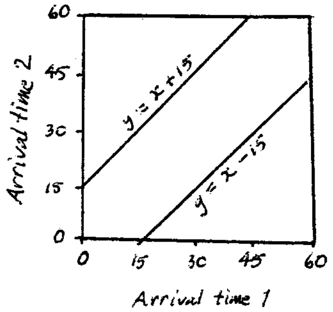
14. All possible times for $T + 70$ minutes, where the minutes value is five times the hour value, are 1:05, 2:10, 3:15, 4:20, 5:25, 6:30, 7:35, 8:40, 9:45, 10:50 and 11:55. Subtracting 70 minutes from these times gives a finite list of possible times T: 11:55, 1:00, 2:05, 3:10, 4:15, 5:20, 6:25, 7:30, 8:35, 9:40 and 10:45. Now consider whether the angle between the hour hand and the minute hand is 45°. Some of the candidates can be eliminated by inspection, but some students forget that the hour hand creeps along as the minute hand moves. To analyze the angle between the two hands of the clock more carefully, represent the time T by $x{:}y$ where x is the value of the hours showing, and y is the value of the minutes, If we measure angles clockwise from vertical, then at time $T = x{:}y$ the minute hand is $\frac{360}{60}y = 6y$ degrees from vertical (since it takes 60 minutes for the minute hand to go 360 degrees), and the hour hand is $\frac{360}{12}\left(x + \frac{1}{60}y\right) = 30x + y/2$ degrees from vertical (it takes 12 hours for the hour hand to go 360 degrees, and the hour hand moves 1/60 of an hour increment for each minute). The angle between the hands is 45°, so $|(30x + y/2) - 6y| = 45$, or $30x - 11y/2 = \pm 45$. The only time from the list that fits the criterion $T = 7{:}30$. A more algebraic solution is possible, but is longer.

15. Draw a diagram. The centres of the circles form an equilateral triangle, and the area enclosed in between the circles can be found by subtracting the areas of the circles inside the triangle from the area of the triangle itself. The radius of each circle is



$r = \sqrt{A/\pi}$ and the angle at each corner of the triangle is 60°, thus the base of the triangle is $b = 2r$ and its height is $h = 2r\cos 60° = r\sqrt{3}$ (alternatively, h can be found by the Pythogarean Theorem). The area of the triangle is $\frac{1}{2}bh = r^2\sqrt{3}$ and the area of each sector contained in the triangle is $\pi r^2 \frac{60}{360} = \frac{1}{6}\pi r^2$. Since there are three of these sectors in the triangle, the area in between the circles is $r^2\sqrt{3} - \frac{1}{2}\pi r^2 = A\left(\frac{\sqrt{3}}{\pi} - \frac{1}{2}\right)$ or approximately $0.0513A$.

16. Suppose we plot the time one friend arrives on the horizonal axis and the time the other friend arrives on the vertical axis. Although students may be unfamiliar with continuous probability distributions, similar two-dimensional diagrams appear in the Math 12 textbook for discrete distributions. Let x be the time in minutes the first friend arrives after 6:00, and let y be the time in minutes the other friend arrives after 6:00. If all times between 6:00 and 7:00 are equally likely for both friends, then all ordered pairs of times (x, y) in the square $0 < x < 60$, $0 < y < 60$ are equally likely. The probability that the two friends arrive within 15 minutes of each other is the ratio of the area of all pairs (x, y) within 15 minutes of each other to the area of the entire square. The times x and y are exactly 15 minutes apart if and only if $|x - y| = 15$, *i.e.* $x - y = \pm 15$ or $y = x \mp 15$, and the point (x, y) represents two times within 15 minutes of each other if the point is between the straight lines $y = x + 15$ and $y = x - 15$. The area between the two straight lines and inside the square is easily calculated by subracting the areas of the two triangles representing pairs of times farther apart than 15 minutes from the area of the square: $60^2 - 45^2 = 1575$. Then the probability the two friends arrive within 15 minutes of each other is $1575/3600 = 0.4375$ or 43.75%.



17. We can use the "guess and check" strategy here, but we should guess intelligently. Consecutive prime numbers are usually reasonably close together, so it is probably not such a bad approximation to assume initially they are all the same. This would give $\sqrt[3]{1113121} \approx 103.64$ which is of course not a prime number. However the three prime numbers we seek can be expected to be near this value. For example, 103 is prime (see below) and $1113121/103 = 10807$ is an integer. The primes immediately above and below 103 are 107 and 101. Check that $101 \times 103 \times 107 = 1113121$, so the three consecutive prime numbers are 101, 103 and 107, and the product of the smallest and largest is $101 \times 107 = 10807$.

    To see if a number greater than 2 is prime, it must be odd, its square root must not be an integer, and it must not be divisible by any prime number less than its square root. These three conditions are enough to conclude that the number is prime (be prepared to explain why). For example to check that 107 is prime, since $\sqrt{107} \approx 10.3$ it suffices to check that 107 is not divisible by 3, 5 or 7.

18. The number 100! is too large for a calculator to display, so a brute force approach will not work with a calculator. $100! = 1 \times 2 \times 3 \times 4 \times \cdots \times 98 \times 99 \times 100$ and it can be seen that every even number in this list of positive integers up to 100 will contribute at least one power of 2. Furthermore, this list contains higher powers of 2: $4 = 2^2$, $8 = 2^3$, $16 = 2^4$, $32 = 2^5$ and $64 = 2^6$; and also some multiples of these numbers, so there are a lot of powers of 2 to count.

To motivate the solution method, a smaller factorial can be considered, say $20! = 1 \times 2 \times \cdots \times 20$. The odd numbers in this list will not contribute any factors of 2. The even numbers in the list are $2 = 1 \times 2^1$, $4 = 1 \times 2^2$, $6 = 3 \times 2^1$, $8 = 1 \times 2^3$, $10 = 5 \times 2^1$, $12 = 3 \times 2^2$, $14 = 7 \times 2^1$, $16 = 1 \times 2^4$, $18 = 9 \times 2^1$ and $20 = 5 \times 2^2$, and there are $1+2+1+3+1+2+1+4+1+2 = 18$ powers of 2 in the list, thus the largest power of 2 that divies 20! is 2^{18}. We rearrange the even numbers as 1×2^1, 3×2^1, 5×2^1, 7×2^1, 9×2^1 (5 odd multiples of 2^1); 1×2^2, 3×2^2, 5×2^2 (3 odd multiples of 2^3); 1×2^3 (1 odd multiple of 2^3); 1×2^4 (1 odd multiple of 2^4). We may notice a pattern. The numbers in the list are numbers less than or equal to 20 that are the odd multiples of powers of 2 less than or equal to 20. The total number of powers of 2 in the list is $5(1) + 3(2) + 1(3) + 1(4) = 18$.

We apply this idea to the original problem. The powers of 2 less than or equal to 100 are 2^1, 2^2, 2^3, 2^4, 2^5 and 2^6. The following information could be organized into a table: the numbers less than or equal to 100 that are odd multiples of 2^1 are 1×2^1, 3×2^1, $\ldots$, 49×2^1 ($(49+1)/2 = 25$ odd multiples of 2^1); those that are odd multiples of 2^2 are 1×2^2, 3×2^2, $\ldots$, 25×2^2 ($(25+1)/2 = 13$ odd multiples of 2^2); those that are odd multiples of 2^3 are 1×2^3, 3×2^3, $\ldots$, 11×2^3 (6 odd multiples of 2^3); those that are odd multiples of 2^4 are 1×2^4, 3×2^4, 5×2^4 (3 odd multiples of 2^4); those that are odd multiples of 2^5 are 1×2^5, 3×2^5 (2 odd multiples of 2^5); those that are odd multiples of 2^6 are 1×2^6 (1 odd multiple of 2^6). Notice that all the odd multiples do not have to be listed to find how many there are, only the largest odd multiple less than or equal to 100. We have accounted for $25 + 13 + 6 + 3 + 2 + 1 = 50$ different even numbers that are less than or equal to 100, which is all of them. The total number of powers of 2 are $25(1) + 13(2) + 6(3) + 3(4) + 2(5) + 1(6) = 97$ and the highest power of 2 that divides 100! is 2^{97}.

The problem could also be solved, somewhat more laboriously, by listing all factors of 100! from 1 to 100, and determining the highest power of 2 that divides each of the one hundred factors. If the problem is done in this way, the student could be asked if any patterns can be detected.