UBC Grade 8–9 Workshop Problems, 2002

1. Billy is 7 years old and has \$21 to spend. He wants to buy a video game on sale for \$50 (half the original price) until Sun. Oct. 28. Billy gets paid his allowance of \$15 every Saturday but insists on depositing 50% of that into a long-term account for his future. If today is Friday Oct. 5, will Billy have enough money to buy his video game?

2. Find 3 consecutive odd integers whose sum is 129. Can the sum equal 127?

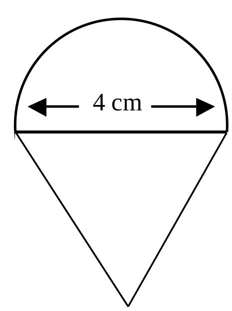
3. Evaluate $200 - 199 + 198 - 197 + 196 - 195 + 194 - \ldots + 2 - 1$.

4. There is a certain number of students in a class. To begin with, the teacher places them in complete rows of seven. Whenever they have to pair up, someone always has to pair with the teacher (ew!). Later on that year a new student joins the class. The teacher can no longer have rows of seven, but instead rearranges them into groups of exactly three students. What was the smallest number of students in the original class?

5. L'il Mac is driving home, a distance of 600 km. He starts his journey at 12:00 noon, travelling at 80 km/h, but increases his speed 10 km/h after every 100 km. If he gets pulled over for speeding when he reaches a speed of 110 km/h, at what time is he stopped?

6. Mr. Ina Rush is in a rush on his way to work. He grabs two socks from his socks drawer at random, so he can put them on while sitting on the bus. Originally (before Mr. Rush grabs any socks) there were ten loose socks in the drawer: 6 black socks and 4 green socks. Five of these ten socks had holes in the toe, and only one green sock did not have any holes. As it turns out, the first sock Mr. Rush picks is a black sock with no holes. Given this, what are the chances Mr. Rush has a matching pair of socks without any holes? (By the way, Mr. Rush forgot his shoes!)

7. In the "ice cream cone" diagram, the triangular cone portion has the same area as the semicircular ice cream portion. What is the total height of the ice cream cone?



8. Brian has a defective 12-hour clock: every 6 minutes, the clock loses one minute. If Brian's clock reads the correct time right now, when will it read the correct time again?

9. Steven has a new glue stick that is 0.02 m in diameter, with an initial volume of 10π cm^3. He spends the weekend putting together a collage and ends up using 300π mm^3 of the glue stick. What is the new height of the glue stick in cm?

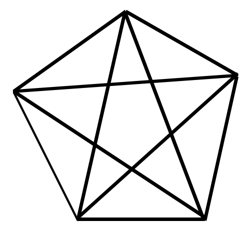
10. Three brothers decide to pool their money in order to buy a new video game system. Allan contributes half as much as his two brothers combined and Bert contributes one-third as much as his two brothers combined. If Charlie contributes \$35 more than Allan, how much does the system cost and how much does each brother contribute?

11. Find the units digit of $1 + 3 + 3^2 + 3^3 + 3^4 + \ldots + 3^{2002}$.

12. John *really* enjoys bowling. The first row of bowling pins has 1 pin, and each row thereafter has 1 more pin than the row before it. He wants to set a record by knocking down 25 rows of pins with just one attempt. How many pins does he have to knock down?

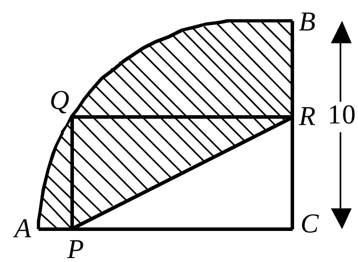
13. Professor Newton had a grade 9 math class with 30 students. At the end of the school year, the class average was 55%. Of the students that passed, their average was 90%, while the students that failed had an average of 48%. How many students will be in Professor Einstein's grade 10 math class the following year?

14. How many different triangles, of various sizes, are there in the shape at the right?



15. Alan enters a judo tournament with the rule that if a contestant loses 2 matches he is thrown out. There are 41 matches, 4 of which are draws, before Alan wins the competition. How many judo players entered the tournament?

16. In the diagram at the right, we are given a quarter circle of radius 10 whose centre is C. Given that the perimeter of the rectangle $QPCR$ is 26, find the perimeter of the shaded region.



17. Neopolitan Bonaparte has three flavours of ice cream in his freezer: vanilla, chocolate, and strawberry. He wants to make himself an ice cream cone with 5 scoops of ice cream (piled one on top of the other). How many different ways can Neopolitan arrange the scoops of ice cream so that (1) no two scoops next to each other on the pile are of the same flavour and (2) there is at least one scoop of every flavour?

18. In the magic square at the right, the sum of the digits in the ith row or column is i times the sum of the digits in the 1st row. Fill in the rest of the square.

		7
4	4	
3		9

UBC Grade 8–9 Workshop Solutions, 2002

1. Billy will get 4 allowances before Oct. 28—on Oct. 6, 13, 20, and 27. From each allowance, he will be able to spend half, or \$7.50. So, Billy will have \$21 + 4 × \$7.50 = \$51 to spend on Oct. 27, and he will be able to buy the video game (assuming there is no tax!).

2. Instead of setting up an equation ($x + (x+2) + (x+4) = 129$), simply note that the middle integer is the average of the three, so it equals 129/3 = 43, and the integers are 41, 43, and 45. The sum cannot equal 127 since it is not a multiple of 3.

3. Group the terms into 100 consecutive pairs each of which has sum 1 to conclude that the answer is 100. The point of this problem is to look for a pattern instead of simply plugging into a calculator.

4. The original number of students must be a multiple of 7 and odd. Also, when 1 is added the number must be a multiple of 3. Write down the numbers that satisfy the first condition: 7, 21, 35, 49,… The third one is the first one that also satisfies the second condition, so the smallest possible number of students is 35.

5. There are 3 legs of 100 km each before L'il Mac is stopped, at 80, 90, and 100 km/h respectively. The total time for the journey is 100/80 + 100/90 + 100/100 = 5/4 + 10/9 + 1 = 3 13/36 hours; since 13/36 * 60 = 130/6 is about 22 minutes, he is stopped at about 3:22 p.m.

6. Look at the drawer after the first sock has been picked: it contains 4 green socks and 5 black socks, a total of 9. All the hole-ly socks are still in the drawer, and there are 5–3=2 black socks with holes (since of the 5 hole-ly socks 3 are green), so 3 of these black socks have no holes. The chances of picking one of these 3 socks from the 9 in the drawer are 3/9 or 1/3. Make sure students understand why it is okay to consider the 9-sock drawer after the first sock has been picked.

7. The ice cream portion has height 2 cm and area $(1/2)\pi(2^2) = 2\pi$ cm^2, so if h is the height of the triangle, we have $(1/2)(4)(h) = 2\pi$ and $h = \pi$ cm; the total height of the cone is $2 + \pi$ cm.

8. The clock runs at 5/6 the correct rate. It will read the correct time again when 1/6 times the number of elapsed hours, which is the amount of time it has lost, is a multiple of 12 (since we have a 12-hour clock); this first happens after 72 hours.

9. We first need to make all measurements consistent in terms of units; since we are asked for an answer in cm, convert to cm: the initial diameter is 2 cm and 0.3π cm^3 of the glue stick is used. The initial height h of the glue stick is given by $\pi r^2 h = 10\pi$, and since $r = 1$ cm, $h = 10$ cm. The new height H is given by $\pi r^2 H = 9.7\pi$, so the new height is $H = 9.7$ cm.

10. Let A, B, C be the amounts contributed by the three brothers. The question can be done as a system of 3 equations in A, B, C. A nicer solution however involves the observation that if T is the total cost, then Allan contributing ½ as much as his brothers combined means $A = (1/3)T$ and similarly $B = (1/4)T$, so $C = (1 – (1/3) – (1/4))T = (5/12)T$. Thus $(5/12)T – (1/3)T = (1/12)T = \$35$ so $T = \$420$; the brothers contribute \$140, \$105, and \$175 respectively.

11. Note that we only need the units digits of the powers of 3: they are 1, 3, 9, 7, 1, 3, 9, 7, …, with a repeating pattern of length 4. Every block of 4 consecutive digits sums to 20, so the sum up to 3^{1999} (500 blocks) has 0 as the units digit. Adding the last 3 terms gives a units digit the same as that of 1 + 3 + 9, so the units digit of the sum is 3.

12. This problem is an excuse for reminding students of the formula $1 + 2 + … + n = n(n+1)/2$. If students have not seen this formula, give a proof: either use Gauss's trick of writing the sum S down backwards to get $2S = n(n+1)$ or give a geometrical argument that rearranges 2 triangles into an n by $n+1$ square. The number of pins is (25)(26)/2 = 325.

13. Let x be the number of students who passed. Then $30 - x$ students failed. Since the students who passed had an average of 90 and those who failed had an average of 48, the sum of the percentage marks of the 30 students is $90x + 48(30 - x)$. But since the overall class average was 55, this equals (55)(30). Solving the equation $90x + 48(30 - x) = (55)(30)$ gives $x = 5$. Only 5 students passed! Considering the total of the marks in the class is the key to this solution; translating the given information on averages into the equation above will not come easily to many students.

14. The keys here are noting that there are big triangles formed by adjoining adjacent regions and finding a systematic way to count *all* of these triangles. One way is to consider 3 cases: 1) The vertices of the triangle are 2 of the exterior 5 vertices—call these A and B—and 1 of the interior 5 vertices: if A and B are adjacent there are 5 choices for the pair A, B and then 3 choices for C, so 15 triangles; if A and B are separated by one (exterior) vertex then there is just one choice for C, so 5 triangles, a total of 20 triangles in this case. 2) The vertices of the triangle are 1 exterior vertex and 2 interior vertices: there are 5 such (all small) triangles. 3) The vertices of the triangle are all 3 exterior vertices: there are 10 such triangles (5 for which the vertices are consecutive and 5 for which one vertex is opposite the other 2 adjacent vertices). So, there are a total of 35 triangles. Students will need guidance in order to do this counting properly; there will be few correct answers.

15. The key is looking at the total number of losses, which is 37. For each contestant who is thrown out there must have been two losses, and since 37/2 = 18.5 it means there were 18 other contestants, or 19 in total. The extra "0.5" comes from the fact that Alan lost one match.

16. Note that the length of PR equals that of QC, since both are diagonals of the rectangle, so PR has length 10. Also, the sum of the lengths of RC and PC is $(1/2)(26) = 13$, so the sum of the lengths of BR and AP is $20 - 13 = 7$. Finally, the arc AB has length $(1/4)(2\pi)(10) = 5\pi$, so the perimeter of the region is $10 + 7 + 5\pi = 17 + 5\pi$. The point here is to solve the problem without solving for the (nonintegral) width and height of the rectangle.

17. There are $3 \times 2 \times 2 \times 2 \times 2 = 48$ arrangements that satisfy (1), since the bottom scoop can be anything but then each subsequent scoop cannot be the same as the one it is put on top of. In these 48 arrangements, if there are at most 2 scoops of each flavour then every flavour is present, so (2) is satisfied. There cannot be 4 scoops of the same flavour since scoops of the same flavour must have another scoop in between, so in the arrangements that satisfy (1) but not (2), scoops 1, 3, and 5 must have the same flavour and scoops 2 and 4 must have a different, common flavour. There are 3 choices for the flavour of scoops 1, 3, and 5 and then two choices for the flavour of scoops 2 and 4. This means $3 \times 2 = 6$ arrangements that satisfy (1) do not satisfy (2), so the number of arrangements for Neopolitan's desert is $48 - 6 = 42$. Students may try to list the possibilities in an ad hoc manner, but it should be stressed that a *systematic* enumeration is necessary in order to ensure that all possibilities are included. Giving a hint: "start by looking at arrangements that satisfy (1) only" may point the students in the right direction.

18. If a and b are the missing entries in the first row, then from the sum of the 1st column equalling that of the 1st row, we have $a + 4 + 3 = a + b + 7$, which gives $b = 0$. Letting c be the middle entry of the 3rd column, and comparing the sum of the 2nd row and 3rd column, which are respectively 2 and 3 times the sum of the 1st row, we have $(1/2)(4 + 4 + c) = (1/3)(7 + c + 9)$, and solving gives $c = 8$. The sum of the 2nd row is now $4 + 4 + 8 = 16$, so the first row sums to 8 and the top-left entry equals 1; the third row sums to 24 so the middle entry of the last row is 12. The entries, by row from left to right are: 1, 0, 7; 4, 4, 8; 3, 12, 9. We can check that all required conditions hold for this magic square. What makes this problem challenging is that none of the row or column sums are initially given, and finding entries involves clever selection of rows or columns to compare.