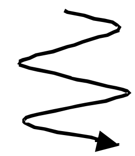
UBC Grade 11–12 Workshop Problems, 2002

1. Brian has a defective 12-hour clock: every 6 minutes, the clock loses one minute. If Brian's clock reads the correct time right now, when will it read the correct time again?

2. Kip skis down Whistler Bowl, which has a slope of 40° relative to the horizontal, in a regular zigzag path as in the diagram at the right. If Kip makes a turn every 5 seconds, takes 11 minutes to reach the bottom, and travels a net distance of 12 m down the slope every two turns, what is the total elevation loss in his run down Whistler Bowl?



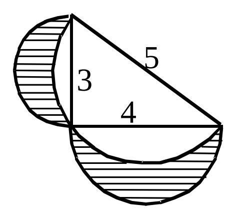
3. If $\log_8 6 = x$, $\log_2 4 = y$, and $\log_{16} 5 = z$, express $\log_2 600$ in terms of x, y, and z.

4. Jim and Macy work together to dig a ditch, and it takes them 4 hrs. It would take Macy 6 hrs more than Jim to dig the ditch alone. How long would it take Jim to dig the ditch alone?

5. Find the units digit of $1 + 3 + 3^2 + 3^3 + 3^4 + \ldots + 3^{2002}$.

6. A swimming pool is 30 m wide, 50 m long, and 7 m deep. After an earthquake, the pool is tilted along one edge and the water covers one side (which was vertical before the earthquake) of the pool completely without any water spilling out. At this point, 3/4 of the base is covered by water. What was the water level in the pool before the earthquake? Does the answer depend on whether the pool was tilted along a 30-m edge or a 50-m edge?

7. A right-angled triangle has sides of length 3, 4, and 5. Semicircles are drawn with these sides as diameters. Explain why the semicircle with the hypotenuse as diameter passes through the other vertex of the triangle, as depicted in the diagram. Find the area of the shaded region.

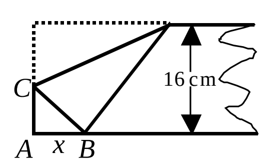


8. Three circles, each with a radius of 4, are all tangent to each other. Determine the perimeter of the equilateral triangle enclosing the circles whose sides are each tangent to two of the circles.

9. Find all positive integers a and b that satisfy the equation $1/a + a/b + 1/ab = 1$.

10. A wheel whose rim has the equation $x^2 + y^2 = 25$ is spinning rapidly counterclockwise. A speck of dirt leaves the rim at the point (3,4) and flies off along the tangent line. How high up does it hit on the wall whose equation is $x = -9$? (Ignore the effect of gravity.)

11. One corner of a long strip of paper 16 cm wide is folded over to the opposite side to make triangle ABC as shown. What is the area of triangle ABC in terms of x, the length of side AB?



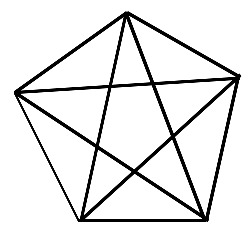
12. Use a calculator to find an *approximate* value of $(3 + 2\cdot2^{1/2})^{1/2} - (3 - 2\cdot2^{1/2})^{1/2}$. The approximate value looks very simple; prove that the *exact* value of this expression is the value suggested by your calculator.

13. Alan and Kika race a single lap around an oval-shaped track. They start from the same place and run in opposite directions. After they pass each other, it takes Kika 16 seconds and Alan 36 seconds to finish the lap. Assuming they each run at constant speed, how long does it take Kika to run the race?

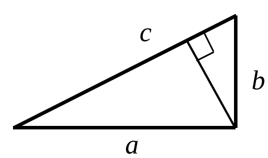
14. A point P lies inside a rectangle so that it is 5 cm from one corner, 14 cm from the opposite corner, and 10 cm from a third corner. How far is P from the fourth corner of the rectangle?

15. A round coin of radius 1 cm is tossed at random onto a floor on which a thin square mat with side length 1 m lies. If some part of the coin lies on the mat, find the probability that the coin lies entirely on the mat.

16. How many different triangles, *of various sizes*, are there in the shape at the right? How can you be sure you have counted *all* the triangles?



17. Using the attached diagram and similar triangles, prove Pythagoras' theorem (if c is the length of the hypotenuse of a right-angled triangle and a and b are the lengths of the other sides, then $a^2 + b^2 = c^2$).



18. Evaluate $\sin 1° \sin 3° \ldots \sin 89°$.

UBC Grade 11–12 Workshop Solutions, 2002

1. The clock runs at 5/6 the correct rate. It will read the correct time again when 1/6 times the number of elapsed hours, which is the amount of time it has lost, is a multiple of 12 (since we have a 12-hour clock); this first happens after 72 hours.

2. Since Kip turns once every 5 seconds, he makes $12 \times 11 = 132$ turns in his descent. There are $132/2 = 66$ pairs of turns, so the total downslope distance is $66 \times 12 = 792$ m. The elevation loss is 792 sin 40°, or about 509 m.

3. $\log_2 600 = \log_2 6\cdot 4\cdot 5^2 = \log_2 6 + \log_2 4 + 2\cdot\log_2 5 = 3\cdot\log_8 6 + \log_2 4 + 2\cdot 4\cdot\log_{16} 5 = 3x + y + 8z$. Students may need to be reminded of basic properties of logarithms.

4. Let x be the time it would take Jim to dig the ditch alone, so Macy's solo time is $x + 6$. The fraction of the ditch dug by Jim in 1 hr is $1/x$, and for Macy it is $1/(x + 6)$; since both together dig the ditch in 4 hours, we have $4/x + 4/(x + 6) = 1$. Now solve the associated quadratic equation $x^2 – 2x – 24 = 0$, or $(x – 6)(x + 4) = 0$; rejecting the negative value we conclude Jim would dig the ditch in 6 hours by himself. As with most "word" problems, the set-up is the key.

5. Note that we only need the units digits of the powers of 3: they are 1, 3, 9, 7, 1, 3, 9, 7, …, with a repeating pattern of length 4. Every block of 4 consecutive digits sums to 20, so the sum up to 3^{1999} (500 blocks) has 0 as the units digit. Adding the last 3 terms gives a units digit the same as that of 1 + 3 + 9, so the units digit of the sum is 3.

6. Pictures are essential. Suppose first that the pool is tilted along the 50 m edge. After the quake, the water has the shape of a wedge 50 m long and whose cross-section is a right angled triangle with base $(3/4)\cdot 30$ m (since only 3/4 of the base is covered) and height 7. The volume of the water is $(1/2)(3/4)\cdot 30\cdot 7\cdot 50$, so the original water level is $((1/2)(3/4)\cdot 30\cdot 7\cdot 50)/(50\cdot 30) = 21/8$ m. The answer is the same if the tilt occurred along the 30 m edge; the symmetry in "50" and "30" in this final answer makes this fact apparent, and in fact these numbers are irrelevant to the final answer. Ask students whether they can explain in more fundamental terms why the answer ends up being 3/8 of 7.

7. The large semicircle passes through the third vertex because the distance from the midpoint of the hypotenuse to this vertex is, as is easily seen from similar triangles, 5/2. Using familiar area formulas, we see that the area inside the large semicircle and outside the triangle is $(1/2)\pi(5/2)^2 – (1/2)\cdot 3\cdot 4$, so the required area is $(1/2)\,)\pi(3/2)^2 + (1/2)\pi(4/2)^2 – ((1/2)\pi(5/2)^2 – (1/2)\cdot 3\cdot 4)$; noting the cancellation of the three circular terms (since $3^2 + 4^2 = 5^2$), we get a final answer of 6. By *not* evaluating the intermediate values but instead using this cancellation, we see that the area, for any right triangle, is the same as the area of the triangle itself!

8. Draw the picture and consider one of the sides. By dropping perpendiculars to this side from the centres of the two circles that are tangent to the side and also joining these centres to the nearest vertex of the side, and noting that we end up with a rectangle and 2 standard 30°-60°-90° triangles, we conclude the length of the side is $8 + 2\cdot 4\cdot 3^{1/2}$, so the triangle's perimeter is triple this, or $24 + 24\cdot 3^{1/2}$. Adding these various lines to the picture is the key; also, it is important to understand *why* those triangles are the standard ones.

9. Expand and solve for b to get $b = a + 1 + 2/(a – 1)$; the integrality of a and b then shows that, since $2/(a – 1)$ is an integer, that a must be 2 or 3. Both of these values of a do give solutions: $a = 2$, $b = 5$ and $a = 3$, $b = 5$. Figuring out how to use the fact that a and b are positive integers is the (non-trivial) key; it is also important to know that *all* solutions have been found.

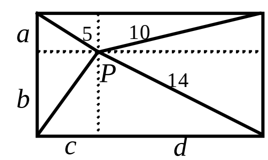
10. Draw diagrams in the Cartesian plane and, using the fact that the tangent to the circle at (3,4) is perpendicular to the radius connecting the centre of the circle to this point, deduce the equation of this tangent line is $y – 4 = (–3/4)(x – 3)$. Let $x = –9$ to conclude the speck of dirt, which travels along this tangent line, hits the wall at height $y = 13$. Introducing the tangent line is the key.

11. Letting y be the height of the triangle, and noting that the line CB and the vertical dashed line have the same length, we arrive at the equation $(x^2 + y^2)^{1/2} = 16 - y$; solving this equation for y gives $y = (16^2 - x^2)/32$, so the area of the triangle is $(16^2x - x^3)/64$.

12. The calculator suggests the answer is 2. There are a couple of ways to prove this: (1) note that since $(2^{1/2} \pm 1)^2 = 3 \pm 2 \cdot 2^{1/2}$, the expression is $(2^{1/2} + 1) - (2^{1/2} - 1) = 2$; (2) square the original expression and note that it becomes $(3 + 2 \cdot 2^{1/2}) + (3 - 2 \cdot 2^{1/2}) - 2 \cdot 1^{1/2} = 4$.

13. Let t be Kika's lap time, so Alan's lap time is $t + (36 - 16) = t + 20$. Letting the unit of distance be a lap, their respective speeds are $1/t$ and $1/(t+20)$. The distances they ran after they passed are $16/t$ and $36/(t + 20)$ respectively, and these distances sum to 1. Solve the resulting quadratic equation $t^2 - 32t - 320 = 0$ to conclude $t = 40$. Alternative solutions that introduce more variables are likely to appear.

14. Refer to the diagram at the right. Drop perpendiculars from P to the sides of the rectangle, and let a, b, c, and d be the lengths of the segments these perpendiculars divide the sides into. By Pythagoras, we have (1) $a^2 + c^2 = 5^2$, (2) $b^2 + d^2 = 14^2$, and (3) $a^2 + d^2 = 10^2$. Subtracting (3) from (1) gives $c^2 - d^2 = 5^2 - 10^2$, and adding this to (2) gives $b^2 + c^2 = 14^2 + 5^2 - 10^2 = 121$, so P is 11 cm from the fourth corner. Ask the students whether the lengths of the sides of the rectangle can be uniquely determined.



15. For the coin to lie entirely on the mat, the centre of the coin must lie in the cocentric central square of side length 98 cm; for the coin to have landed so that some part of it is on the mat, the centre of the coin must lie in the figure obtained by replacing each corner of the cocentric square of side length 102 cm by a quarter circular arc of radius 1 cm. (These facts require some pictures and insight to see.) The smaller square has area 98^2 cm^2, and the large rounded "square" has area $100^2 + 4 \cdot 100 + \pi$ cm^2, so the required probability is the quotient of the former to the latter, which is approximately .923, so the probability is about 92.3%.

16. The keys here are noting that there are big triangles formed by adjoining adjacent regions and finding a systematic way to count *all* of these triangles. One way is to consider 3 cases: 1) The vertices of the triangle are 2 of the exterior 5 vertices—call these A and B—and 1 of the interior 5 vertices: if A and B are adjacent there are 5 choices for the pair A, B and then 3 choices for C, so 15 triangles; if A and B are separated by one (exterior) vertex then there is just one choice for C, so 5 triangles, a total of 20 triangles in this case. 2) The vertices of the triangle are 1 exterior vertex and 2 interior vertices: there are 5 such (all small) triangles. 3) The vertices of the triangle are all 3 exterior vertices: there are 10 such triangles (5 for which the vertices are consecutive and 5 for which one vertex is opposite the other 2 adjacent vertices). So, there are a total of 35 triangles. Students will need guidance in order to do this counting properly; there will be few correct answers.

17. Let c_1 and c_2 be the lengths of the line segments the perpendicular divides the hypotenuse into (going from left to right). Noting that the two small triangles in the diagram are both similar to the large one, we have $c_1/a = a/c$ and $c/b = b/c_2$. These equations give $a^2 = c_1c$ and $b^2 = c_2c$; adding gives $a^2 + b^2 = (c_1 + c_2)c = c^2$, as required. Alternatively, use the notion of scaling: since the three triangles are similar and areas of similar regions are proportional to the squares of their linear dimensions, the fact that the two small triangles have total area equal to that of the large triangle and the linear proportionality ratio of the three triangles is a:b:c, we conclude $a^2 + b^2 = c^2$. These short proofs could lead to a discussion of the many alternative proofs of Pythagoras' theorem or to a discussion on other Diophantine equations.

18. Multiply and divide by sin 2° sin 4° … sin 88° and write the numerator as (sin 1° sin 89°)(sin 2° sin 88°) … (sin 44° sin 46°)(sin 45°) = (sin 1° cos 1°)(sin 2° cos 2°) … (sin 44° cos 44°)(sin 45°) = ((1/2)sin 2°)((1/2)sin 4°) … ((1/2)sin 88°)(sin 45°). Cancelling most of the sin's with the corresponding ones in the denominator gives us a final answer of 2^{-44} sin 45° = $2^{-44.5}$.