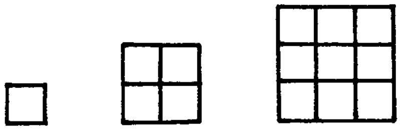
UBC Grade 8–9 Problems 2001

1. How many squares, of various sizes, are in the pictures below? How many would be in a chessboard (an 8×8 picture)?



2. In order to reach the "hot seat" on the show "Who Wants To Be A Zillionaire?" a contestant must answer a "fastest finger question". The question gives you five possible answers A, B, C, D, E which must be arranged into the correct order. If you just guess at random, what chance do you have to reach the hot seat?

3. In a soccer tournament, 15 teams must play every other team exactly once. How many games need to be played?

4. A fat hose would fill a swimming pool in 12 hours by itself. A medium hose would fill the same pool in 18 hours, and a thin hose would fill it in 36 hours. If all three hoses are used at the same time, how long would it take to fill the pool?

5. A cylindrical can with diameter 7 cm and height 12 cm is filled with juice concentrate. The concentrate is emptied into a cylindrical glass container with diameter 12 cm, then the can is filled three times with water and emptied each time into the glass container. How high does the mixture reach up the side of the glass container (assuming the container is tall enough to contain all the mixture)?

6. Grapes are 89% water, and raisins are only 42% water. (a) How many kilograms of grapes are needed to make 1 kg of raisins? (b) How many kilograms of raisins are obtained from 10 kg of grapes?

7. What is the 1057th number in the sequence 7, 2, 15, 10, 5, 0, 7, 2, 15, 10, 5, 0, 7, 2, 15, 10, 5, 0, ...?

8. Suppose you have an even number of quarters. If you put them in groups of five, none are left over, but if you put them in groups of eleven, nine are left over. How many quarters could there be?

9. Some 1 cm cubes are glued together to construct the sides and bottom of an open box. If the outside dimensions of the box are 10 cm $\times$ 10 cm $\times$ 10 cm and the sides and bottom are 1 cm thick, what is the number of cubes used?

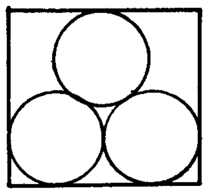
10. Three bags of flour and two bags of sugar weigh 15.436 kg. Four bags of flour and three bags of sugar weigh 20.884 kg. If all the bags of flour weigh the same and all the bags of sugar weigh the same, how much does two bags of flour and one bag of sugar weigh?

11. Sam has a 56% average on his first seven exams. If all his exams are marked out of 100, what would Sam have to get on his eighth exam to get an average of 60% overall?

12. A square is divided into 7 equal rectangles (horizontal strips). If the perimeter of each rectangle is 32 cm, what is the perimeter of the square?

13. A videotape can record 2 hours in SP mode, 4 hours in LP mode, or 6 hours in EP mode. If Jenny records 28 minutes of "Friends" in SP mode and 57 minutes of "Who Wants To Be A Millionaire?" in EP mode, how many minutes of recording time does she have left in LP mode?

14. Three identical basketballs each with circumference 77 cm just fit into a rectangular box, touching each other and the sides of the box as shown below (in a top view), without distorting the balls or the box. What is the volume of the box if its flat lid just touches the tops of the three basketballs?



15. The circumference of the earth at the equator is 36 000 km. A satellite is orbitting the earth so that it is always 600 km above a specific point on the equator. How fast, in km/h, is the satellite moving?

16. The irrational number 0.12340123400123400001234... is formed by using alternating blocks of 1234 and zeros, where the nth block of zeros following the decimal contains n zeros. What is the digit in the 2550th place following the decimal?

UBC Grade 8–9 Solutions 2001

1. In the first picture there is one 1×1 square, in the second picture there is the one 2×2 square plus four 1×1 squares for a total of $1 + 4$, in the third picture one 3×3 square, four 2×2 squares and nine 1×1 squares for a total of $1+4+9$. If the students recognize the pattern rather than (or in addition to) just counting squares, they may be able to guess that a 4×4 picture would have $1+4+9+16$ squares of various sizes. A chessboard would have $1+4+9+16+25+36+49+64 = 204$ squares. (Particularly mathematically minded students could think about how to prove this.)

2. First find all the ways the answers could be arranged. A tree diagram would help the explanation. There are 5 possible ways to choose the answer to the first question. For each choice of answer for the first question there are 4 possible ways to choose the answer for the second question. The students need to be convinced that to find how many ways there are to choose answers to the first two questions, you multiply 5×4. This could be done by systematically listing some examples and counting, until the students see the pattern. Continuing, there are 3 possible ways to answer the third question after the first two have been answered, 2 ways to answer the fourth question after the first three have been answered, leaving only one way to answer the last question. So the total number of different ways to answer the five questions is $5 \times 4 \times 3 \times 2 \times 1 = 120$. Only one of these ways is correct, so choosing at random gives you a 1 in 120 chance to reach the hot seat.

3. Line up the teams. Team 1 must play each of the 14 remaining teams, so this accounts for 14 games. Then team 2 already has a game with team 1, so we need 13 more games, for team 2 to play teams 3 to 15. Team 3 already has games with teams 1 and 2, so we need 12 games with teams 4 to 15. By this time the students hopefully will see a pattern. We need $14+13+12+11+10+9+8+7+6+5+4+3+2+1 = 105$ games.

   This problem could be used as an excuse to develop a general formula, useful for the last question. Adding

$$\begin{array}{ccccccccccccc} & 1 & + & 2 & + & 3 & + & \cdots & + & 12 & + & 13 & + & 14 \\ + & 14 & + & 13 & + & 12 & + & \cdots & + & 3 & + & 2 & + & 1 \end{array}$$

   we get $(1+14)+(2+13)+(3+12)+\cdots+(12+3)+(13+2)+(14+1) = 14 \times 15$, but this is twice the desired sum, so $1+2+3+\cdots+14 = (14 \times 15)/2 = 105$. Similarly for any positive integer n, we have $1+2+3+\cdots+n = n(n+1)/2$.

4. In one hour, the fat hose would fill $\frac{1}{12}$ of the pool by itself, the medium hose would fill $\frac{1}{18}$ of the pool, and the thin hose would fill $\frac{1}{36}$. In one hour, all three hoses together would fill $\frac{1}{12} + \frac{1}{18} + \frac{1}{36} = \frac{1}{6}$ of the pool, so it would take 6 hours to fill the entire pool.

5. The volume of the mixture is the same as the volume of four full cans, $V = 4\pi(3.5)^2(12) = 588\pi$ cm^3. The mixture in the glass container forms a cylinder with radius 6 and height h, with volume $\pi(6)^2h$. Equating $588\pi = \pi(6)^2h$ and solving for h, we get $h = 588/36 = 16\frac{1}{3}$ cm.

6. (a) If we could remove *all* the water from the 1 kg of raisins, we would be left with a pile of dry material with a mass of 58% of 1 kg, or 0.58 kg. This dry material was 11% of the grapes, so we need $0.58/0.11 \approx 5.273$ kg of grapes to make 1 kg of raisins. (b) From part (a) 1 kg of raisins are obtained from $58/11$ kg of grapes, so $10(11/58) \approx 1.897$ kg of raisins are obtained from 10 kg of grapes.

7. The sequence repeats after every 6 numbers, so the 6th, 12th, 18th, etc. number is a 0. The nth number is a 0 if n is an integer multiple of 6. $1057/6 \approx 176.2$ so the largest integer multiple of 6 that is less than 1057 is $6 \times 176 = 1056$, therefore the 1056th number is a 0. The next number in the sequence is the 1057th, which is a 7.

8. The number of quarters is even, so is divisible by 2. The number is also divisible by 5, so must be divisible by $2 \times 5 = 10$. We have narrowed down our search of possible numbers of quarters to 10, 20, 30, ... If we subtracted 9 from the number of quarters, the resulting number would be divisible by 11. But if we subtract 9 from a number divisible by 10, the last digit must be a 1. So what numbers whose last digits are 1 are divisible by 11? Only 11 times some number whose last digit is 1: $11 \times 1 = 11$, $11 \times 11 = 121$, $11 \times 21 = 231$, $11 \times 31 = 341$, etc. Adding 9 to each of these gives the possible numbers of quarters: 20, 130, 240, 350, ... (an arithmetic sequence).

9. A drawing, or even a physical model would be helpful. The base of the box uses $10 \times 10 = 100$ of the cubes, the front and back of the box then each take $10 \times 9 = 90$ cubes, and the left and right sides of the box each taking $9 \times 8 = 72$ cubes, for a total of $100 + 2(90) + 2(72) = 424$ cubes. Many alternative ways exist to divide up the sides. For example, a more symmetric way is to consider the 10×10 base, and four 9×9 sides (draw a picture to show how they fit together).

10. This solution could be illustrated with pictures to avoid algebra. The difference between the two given configurations is one bag of flour and one bag of sugar, which therefore must weigh $20.884 - 15.436 = 5.448$ kg. To answer the question, we now need to find the weight of one additional bag of flour. But two bags of flour and two bags of sugar must weigh $2(5.448) = 10.896$ kg, and comparing this with the first statement in the question we find that one bag of flour must weigh $15.436 - 10.896 = 4.540$ kg, so two bags of flour and one bag of sugar weigh $4.540 + 5.448 = 9.988$ kg.

11. If Sam's total score on his first seven exams is divided by 700, the result is 56% or 0.56, so this total is $(0.56)(700) = 392$. To get 60% over eight exams, the score S on the eighth exam must be added to the total for the seven previous exams, and the new total when divided by 800 must give 60% or 0.6: $(392 + S)/800 = 0.6$. Solving for S gives 88.

12. If we call x the height, in cm, of each horizontal strip, then the width of the strip (and the length of each side of the square) is $7x$ cm and the perimeter is $2x + 14x = 32$ cm. Solving for x gives 2, so each side of the square is $7(2) = 14$ cm long. The perimeter is $4(14) = 56$ cm.

13. It is convenient to convert the playing times all to the equivalent times in LP mode, since the question asks how much time is left in LP mode. Two hours in SP mode takes the same amount of tape as four hours in LP, so at this rate 28 min in SP is equivalent to $28(4/2) = 56$ min in LP. On the other hand, six hours in EP mode takes the same amount of tape as four hours in LP, so 57 min in EP is equivalent to $57(4/6) = 38$ min in LP. There are $4(60) = 240$ min total in LP mode on the tape, so $240 - 56 - 38 = 146$ min are left to record in LP.

14. The centres of the basketballs form the corners of an equilateral triangle, and the length of a side of this triangle is equal to two basketball radii, i.e. one basketball diameter, $77/\pi$ cm. The length of the box is the base of the triangle plus two radii,

or $2(77/\pi) = 154/\pi$ cm, and the width of the box is the height of the triangle plus two radii. By Pythagoras' Theorem the height of the triangle is $77\sqrt{3}/(2\pi)$ cm, so the width of the box is $77(1+\sqrt{3}/2)/\pi$ cm. The height of the box is one basketball diameter $77/\pi$ cm, so the volume is the product of the length, width and height: $77^3(2+\sqrt{3})/\pi^3 \approx 54\,950$ cm^3.

15. We first find the distance from the satellite to the centre of the earth, which defines the radius of a circle. Then the speed of the satellite can be calculated since the satellite travels around the circumference of this circle in 24 hours. The radius of the earth is $36\,000/(2\pi) = 18\,000/\pi$ km, so the satellite is $600 + 18\,000/\pi$ km from the centre of the earth (approximately 6 330 km). The circle that the satellite travels on has circumference $2\,\pi(600 + 18\,000/\pi) = 1\,200\,\pi + 36\,000 \approx 39\,770$ km. This distance is travelled in 24 hours, so the speed is $(1\,200\,\pi + 36\,000)/24 \approx 1\,657$ km/h.

16. Find the place of the last zero in last complete block of zeros in the sequence before the 2550th place, then count the remaining places to find the digit in the 2550th place. The last zero in the 1st block of zeros is in place number $4+1=5$; the last zero in the 2nd block of zeros is in place number $4+1+4+2 = 4(2)+(1+2) = 11$; the last zero in the 3rd block of zeros is in place number $4+1+4+2+4+3 = 4(3)+(1+2+3)$. In general the last zero in the nth block of zeros is in place number $4n+(1+2+\cdots+n)$. Using the result derived in the solution to question 3, the last expression can be more conveniently written as $4n+n(n+1)/2$. Now we want to find the largest n so that $4n+n(n+1)/2 \leq 2550$, to find the position of the end of the last complete block of zeros before the 2550th place. This could be done using the quadratic formula, but most Grade 8–9 students would not have learned this yet. A systematic search could be used instead. For example, something like the bisection method could be used: $n = 50$ gives 1475 (the last zero in the 50th block of zeros is in the 1475th place), $n = 100$ gives 5450. Since 2550 is between 1475 and 5450, try $n = 75$, which gives 3150, so the next place to look is between $n = 50$ and $n = 75$. Eventually one finds that $n = 67$ gives 2546 (and $n = 68$ gives 2618 which is larger than 2550), which means that the last zero in the 67th block of zeros is in the 2546th place. Then the digit in the 2547th place is a 1, the digit in the 2548th place is a 2, the digit in the 2549th place is a 3, and finally the digit in the 2550th place is a 4.