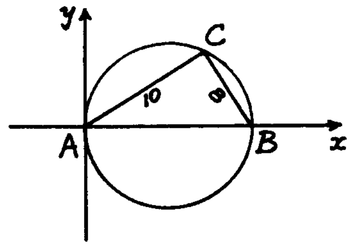
UBC Math 11/12 Problems 2001

1. In order to reach the "hot seat" on the show "Who Wants To Be A Zillionaire?" a contestant must answer a "fastest finger question". The question gives you five possible answers A, B, C, D, E which must be arranged into the correct order. If you just guess at random, what chance do you have to reach the hot seat?

2. In a soccer tournament, 15 teams must play every other team exactly once. How many games need to be played?

3. In the picture below, AB is a diameter of the circle. If $AC = 10$ and $BC = 8$, find the equation of the circle.



4. A father and son run a 15 km race. The father runs one and a half times as fast as his son and crosses the finish line 45 minutes before his son. Find the speeds of the father and the son in km/h, if the speeds are constant, and find how long, in hours and minutes, they each take to run the race.

5. Grapes are 89% water, and raisins are only 42% water. (a) How many kilograms of grapes are needed to make 1 kg of raisins? (b) How many kilograms of raisins are obtained from 10 kg of grapes?

6. Suppose you have an even number of quarters. If you put them in groups of five, none are left over, but if you put them in groups of eleven, nine are left over. How many quarters could there be?

7. A window washer is given the job to wash the exterior of an office building whose sides are entirely covered with glass. The building is a rectangular prism with a square roof of dimensions 10×10 metres and is 80 metres high. The window washer will be anchored to the centre of the roof (which is not glass) by a rope which will allow her to reach any distance up to 10 metres from the anchor point. If the window washer gets paid \$10 per square metre of cleaned window, how much will her paycheque be?

8. A room has a floor that is made up of square tiles all of the same size, laid side by side to form a rectangle 315 tiles wide and 405 tiles long. If a straight line is drawn diagonally from one corner of the floor to the opposite corner, how many tiles will the line intersect?

9. There are eight tangent lines equally spaced around the outside of a circle of radius 3 cm. What is the area outside the circle and inside the octagon formed by the tangent lines?

10. Find all values of x that satisfy $\log_5(2x+1) = 1 - \log_5(x+2)$.

11. A group of businesses pledged a total of \$3000 per year for the upkeep of a pedestrian mall. They intended to split the cost equally, but two of them went out of business before the money was collected. If the remaining businesses split the \$3000 equally, they would each have to pay \$50 more than they originally intended. How may businesses were in the original group?

12. A piece of wire x cm long is bent into a square. For what lengths of the wire will the perimeter (in cm) of the square be greater than the area (in cm^2) of the square?

13. The irrational number 0.12340123400123400001234... is formed by using alternating blocks of 1234 and zeros, where the nth block of zeros following the decimal contains n zeros. What is the digit in the 2550th place following the decimal?

14. Carmen's snoopy sisters like to steal her personal diary and read it for entertainment. So Carmen decides to write her next entry in code, using the equation

$$(\text{code }\#) = 4(\text{letter }\#) - 3,$$

where the "letter #" for A is -39, for B is -36, for C is -33, ..., for Z is 36. What does $(-123)(-111)(-159)(45)\ (-123)(-63)(-159)(45)(129)$ say?

15. Magda likes to go for rides on her mountain bike. She keeps the chain on her 36-tooth front chainring and rides for 2 km with the chain on her 12-tooth rear sprocket, 4 km on her 24-tooth sprocket and 3.5 km on her 18-tooth sprocket. If Magda's wheel is 66 cm in diameter and she keeps pedalling at 80 revolutions per minute, how long, to the nearest minute, does the ride take? How fast, in km/h, does she go in each part of her ride?

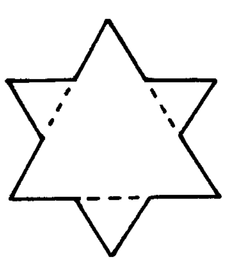
16. Colin is operating a tank on a flat battlefield, and he wants to hit a target on the battlefield 500 metres away. When the tank fires its cannon, the shell leaves at a speed of 85 metres per second. If x represents the horizontal distance in metres from the tank in the direction of the target and y represents the altitude in metres above the battlefield, and if A is the angle above the horizontal that the cannon is pointed, then at time t in seconds after the cannon is fired, the position of the shell is given by

$$x(t) = 85t\cos A, \qquad y(t) = 85t\sin A - 4.9t^2.$$

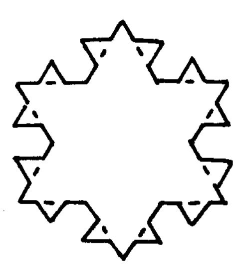
(a) How should Colin angle the cannon in order to hit the target? (b) When does the shell reach its highest point, and how high is it then? (c) What if the target was 5 km away?

17. Three companies offer you a job. Alexor offers a salary of \$70 000 per year, while Bantek promises \$65 000 per year to start, with a salary increase of 4% a year. Calico will pay you initially at a rate of \$60 000 per year, with an increase of 2% every quarter year. If you expect to stay with the same company for five years, where would you make the most money overall?

18. A Koch snowflake is constructed by starting with an equilateral triangle with side length ℓ. For Step 1, subdivide each side of the perimeter into three segments of equal length, remove the middle segment and replace it by a smaller equilateral triangle pointing outwards whose base coincides with the removed middle segment. The resulting figure after Step 1 is a polygon with twelve sides of equal length $\ell/3$, as shown below. Suppose this procedure is repeated infinitely many times: For Step n, subdivide each side of the perimeter of the figure after the previous step into three segments of equal length, remove the middle segment and replace it by a smaller equilateral triangle pointing outwards whose base coincides with the removed middle segment. For example, the resulting figure after Step 2 is a polygon with forty-eight sides of equal length $\ell/9$. The limiting figure as $n \to \infty$ is called a Koch snowflake. What is the area of the Koch snowflake?



After Step 1



After Step 2

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UBC Math 11/12 Solutions 2001

1. First find all the ways the answers could be arranged. A tree diagram can help the explanation. There are 5 possible ways to choose the answer to the first question. For each choice of answer for the first question there are 4 possible ways to choose the answer for the second question. The students need to be convinced that to find how many ways there are to choose answers to the first two questions, you multiply 5×4. This could be done by systematically listing some examples and counting, until the students see the pattern. Continuing, there are 3 possible ways to answer the third question after the first two have been answered, 2 ways to answer the fourth question after the first three have been answered, leaving only one way to answer the last question. So the total number of different ways to answer the five questions is $5 \times 4 \times 3 \times 2 \times 1 = 120$. Only one of these ways is correct, so choosing at random gives you a 1 in 120 chance to reach the hot seat.

2. Line up the teams. Team 1 must play each of the 14 remaining teams, so this accounts for 14 games. Then team 2 already has a game with team 1, so we need 13 more games, for team 2 to play teams 3 to 15. Team 3 already has games with teams 1 and 2, so we need 12 games with teams 4 to 15. By this time the students may see a pattern. We need $14+13+12+11+10+9+8+7+6+5+4+3+2+1 = 105$ games.

   This problem could be used as an excuse to develop a general formula, useful for one of the later questions. Adding

$$\begin{array}{ccccccccccccc} & 1 & + & 2 & + & 3 & + & \cdots & + & 12 & + & 13 & + & 14 \\ + & 14 & + & 13 & + & 12 & + & \cdots & + & 3 & + & 2 & + & 1 \end{array}$$

   we get $(1+14)+(2+13)+(3+12)+\cdots+(12+3)+(13+2)+(14+1) = 14 \times 15$, but this is twice the desired sum, so $1+2+3+\cdots+14 = (14 \times 15)/2 = 105$. Similarly for any positive integer n, we have $1+2+3+\cdots+n = n(n+1)/2$.

3. We determine the radius of the circle, then locate the centre and find the equation. Since the chord AB subtending $\angle ACB$ is a diameter, the $\angle ACB = 90°$, and $\triangle ABC$ is a right triangle with hypotenuse AB. By the Pythogarean Theorem, $AB = \sqrt{(AC)^2 + (CB)^2} = \sqrt{164} = 2\sqrt{41}$. It follows that the radius of the circle is $\frac{1}{2}AB = \sqrt{41}$, the centre has coordinates $(\sqrt{41}, 0)$, and the circle has equation $(x-\sqrt{41})^2 + y^2 = 41$.

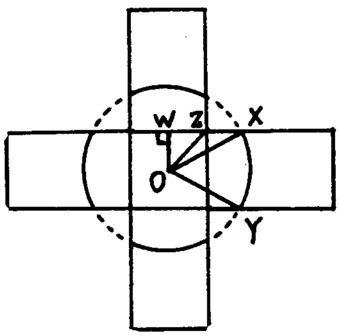
4. Recall that the time taken to travel a distance at a constant speed (rate of travel) is the distance divided by the speed. Both father and son travel the same distance, 15 km. Let s denote the speed of the son in km/h, so the time he takes to run the race is $\frac{15}{s}$ hours. Then the father's speed is $\frac{3s}{2}$, and the time he takes to run the race is $15/(\frac{3s}{2}) = \frac{10}{s}$ hours. But we also know that the son takes 3/4 of an hour longer to travel the distance, so $\frac{15}{s} = \frac{10}{s} + \frac{3}{4}$. Solving for s gives the speed of the son, $\frac{20}{3} = 6\frac{2}{3}$ km/h, and the time he takes to run the race is $15\left(\frac{3}{20}\right) = \frac{9}{4} = 2\frac{1}{4}$ hours, or 2 hours and 15 minutes. The speed of the father is $\frac{3}{2}\left(\frac{20}{3}\right) = 10$ km/h, and his time is $10\left(\frac{3}{20}\right) = \frac{3}{2} = 1\frac{1}{2}$ hours, or 1 hour and 30 minutes.

5. (a) If we could remove *all* the water from the 1 kg of raisins, we would be left with a pile of dry material with a mass of $100\% - 42\% = 58\%$ of 1 kg, or 0.58 kg. This dry

material is $100\% - 89\% = 11\%$ of the grapes, so we need $0.58/0.11 \approx 5.273$ kg of grapes to make 1 kg of raisins. (b) From part (a) 1 kg of raisins are obtained from $58/11$ kg of grapes, so $10(11/58) \approx 1.897$ kg of raisins are obtained from 10 kg of grapes.

6. The number of quarters is even, so is divisible by 2. The number is also divisible by 5, so must be divisible by $2 \times 5 = 10$. We have narrowed down our search of possible numbers of quarters to 10, 20, 30, ... If we subtracted 9 from the number of quarters, the resulting number would be divisible by 11. But if we subtract 9 from a number divisible by 10, the last digit must be a 1. So what numbers whose last digits are 1 are divisible by 11? Only 11 times some number whose last digit is 1: $11 \times 1 = 11$, $11 \times 11 = 121$, $11 \times 21 = 231$, $11 \times 31 = 341$, etc. Adding 9 to each of these gives the possible numbers of quarters: 20, 130, 240, 350, ... (an arithmetic sequence).

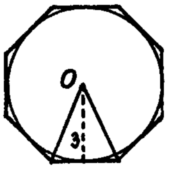
7. We first find the surface area of the building that is accessible to the window washer. Once we have this, we can multiply it by the wage rate to find the amount paid. To see the surface area we can view the building from above and flatten it out as shown. The accessible area is equal to the area of the circle of radius 10 m with centre O at the centre of the roof, intersected with the building sides. It is convenient to initially consider the sides and roof together, then subtract the area of the roof. We can also take advantage of the symmetries of the building. Let us find the area of the circular sector XOY first. Since $OX = OY = XY = 10$ m, the angle $\angle XOY = 60°$, therefore the sector XOY is one-sixth of a circle, and has area $\frac{1}{6}(\pi 10^2) = \frac{50\pi}{3}$ m^2. Determining the area of $\triangle XOZ$ is a little more difficult. Its height is $OW = 5$ m, and its base is $XZ = XW - ZW$. Clearly $ZW = 5$ m, and by the Pythogarean theorem $XW = \sqrt{OW^2 + OX^2} = 5\sqrt{3}$ m, so $XZ = 5(\sqrt{3}-1)$ m. Therefore the area of $\triangle XOZ$ is $25(\sqrt{3}-1)/2$ m^2. Using symmetry, the area of the circle intersected with the roof and sides is $4\left(\frac{50\pi}{3}\right) + 8\left(\frac{25(\sqrt{3}-1)}{2}\right) \approx 282.644$ m^2. Subtracting the area of the 10 m $\times$ 10 m roof leaves 182.644 m^2, the area of glass cleaned. Finally, multiplying by the rate of pay (\$10 per m^2) gives the amount of the paycheque, \$1 826.44. Notice that the height of the building, as long as it is at least 5 m, does not affect the amount of the paycheque.



8. We can first simplify the problem by dividing both 315 and 405 by their common factor 45, and consider a rectangle 7 tiles wide and 9 tiles long. (Use a diagram and some discussion to show that this is a valid simplification.) Then a careful diagram of a 7×9 grid is helpful. Starting in one corner, a diagonal line to the opposite corner has slope $\frac{7}{9}$. Since 7 and 9 are relatively prime, the diagonal can intersect no corners of tiles between the endpoints: the intersections with tile edges occur at coordinates $(0,0), \left(1, \frac{7}{9}\right), \left(2, \frac{14}{9}\right), \cdots, \left(8, \frac{56}{9}\right), (9,7)$, i.e. at $(0,0), \left(1, \frac{7}{9}\right), \left(2, 2\frac{1}{3}\right), \cdots, \left(8, 6\frac{2}{9}\right), (9,7)$.

These coordinates can be used to accurately plot the diagonal line. From the accurate plot we can see that 15 square tiles are intersected, and for the entire floor $15 \times 45 = 675$ tiles are intersected.

9. Since the eight tangent lines are equally spaced, they form a regular octagon. We find the area inside the octagon, then subtract the area inside the circle. The area inside the octagon can be found by subdividing the octagon into eight isosceles triangles of equal size, each with angle $360°/8 = 45°$ at O. Each of these eight triangles can be further subdivided into two right triangles with side 3 cm and angle $45°/2 = 22.5°$ at O. Then the area of each right triangle is $\frac{1}{2} \cdot 3 \cdot 3\tan(22.5°)$ cm^2, and the area inside the octagon is $16 \cdot \frac{1}{2} \cdot 3 \cdot 3\tan(22.5°)$ cm^2. Thus the area inside the octagon and outside the circle is $72\tan(22.5°) - 9\pi \approx 1.55$ cm^2.



10. Using basic properties of logarithms, we obtain $\log_5(2x+1) + \log_5(x+2) = \log_5(2x+1)(x+2) = 1$. Then taking the exponential function (with base 5) we get $(2x+1)(x+2) = 5^1$. Solving the quadratic equation $(2x+1)(x+2) - 5 = 2x^2 + 5x - 3 = 0$ (by factoring, or using the quadratic formula) gives solutions $x = -3$ or $x = \frac{1}{2}$. However both $2x+1$ and $x+2$ must be positive for the logarithms in the original problem to be defined, therefore only $x = \frac{1}{2}$ satisfies the relation.

11. We can use algebra to solve this problem. Let n be the original number of businesses in the group, so $n-2$ businesses remain. Originally the share for each business is $\$3000/n$, but after two went out of business the share is $\$3000/(n-2)$. We are told that $\frac{\$3000}{n-2} = \$50 + \frac{\$3000}{n}$, which after some algebraic manipulation reduces to $n(n-2) - 120 = n^2 - 2n - 120 = 0$. Solving this quadratic equation by factoring or using the quadratic formula gives $n = 12$ or $n = -10$. Since the number of businesses must be positive, we reject $n = -10$: there were 12 businesses in the original group. (Some students may guess this, but they could be asked if they can prove this is the only possible solution.)

12. We use algebra and obtain a quadratic inequality. If x is the perimeter of the square, then $x/4$ is the length of each side, thus the area is $x^2/16$ cm^2. For the perimeter to be greater than the area we want $x > x^2/16$, or $x(x-16) < 0$. This shows that x and $x-16$ must have opposite signs. But x cannot be negative so we need $x > 0$ and $x - 16 < 0$, therefore x must be strictly greater than 0 cm and strictly less then 16 cm. (This can also be seen by plotting the parabola $y = x(x-16)$.)

13. Find the place of the last zero in last complete block of zeros in the sequence before the 2550th place, then count the remaining places to find the digit in the 2550th place. The last zero in the 1st block of zeros is in place number $4+1=5$; the last zero in the 2nd block of zeros is in place number $4+1+4+2 = 4(2)+(1+2) = 11$; the last zero in the 3rd block of zeros is in place number $4+1+4+2+4+3 = 4(3)+(1+2+3)$. It is hopefully clear that the last zero in the nth block of zeros is in place number

$4n + (1 + 2 + \cdots + n)$. Using the result derived in the solution to question 3, the last expression can be more conveniently written as $4n + n(n+1)/2$. Now we want to find the largest n so that $4n + n(n+1)/2 \leq 2550$, to find the position of the end of the last complete block of zeros before the 2550th place. This could be done using the quadratic formula: solving $4n + n(n+1)/2 - 2550 = 0$ gives $n \approx 67.06$ or -152.11. But n should be a positive integer, and no larger than 67.06. Then $n = 67$ gives $4n + n(n+1)/2 = 2546$ (and $n = 68$ gives 2618 which is larger than 2550), which means that the last zero in the 67th block of zeros is in the 2546th place. This implies that the digit in the 2547th place is a 1, the digit in the 2548th place is a 2, the digit in the 2549th place is a 3, and finally the digit in the 2550th place is a 4. (A systematic search for n could be used instead.)

14. A sample calculation may be needed to help people understand the code. For example the code # for the letter A is $4 \cdot (-39) - 3 = -159$, so (-159) represents the letter A. If n represents the position of the letter in the alphabet ($n = 1$ for the letter A, $n = 2$ for the letter B, $n = 3$ for the letter C, etc.) then the "letter #" m is given by $m = 3n - 42$. Therefore the "code #" ℓ satisfies $\ell = 4m - 3 = 4(3n-42) - 3 = 12n - 171$. Inverting this last expression (solving for n in terms of ℓ) gives $n = (\ell + 171)/12$. It is now straightforward to decode the message: $\ell = -123$ corresponds to $n = 4$ or the letter D, $\ell = -111$ corresponds to $n = 5$ or the letter E, etc. The message is DEAR DIARY. (This could also be done by first making a chart for the entire alphabet, a longer method of solution for this short message.)

15. The time taken to travel a distance at a constant speed is the distance divided by the speed. In this problem the speed must be calculated using the gear ratio, the circumference of the wheel, and the rate of pedalling. In the first part of the ride, the gear ratio is 3:1. The rear wheel, which is attached to the rear sprocket, goes around three times for every one time the pedals go around (the pedals are attached to the front chainring). Therefore, for every revolution of the pedals, the rear wheel travels three circumferences (assuming the wheel does not slip), or $3 \cdot 0.66\pi$ m. There are 80 revolutions per minute, or $80 \cdot 60$ revolutions per hour, so Magda travels at $80 \cdot 60 \cdot 3 \cdot 0.66\pi/1000 \approx 29.9$ km/h. To travel 2 km at this rate takes approximately $2/29.9 = 0.067$ hours, or about 4 min. In the second part of the ride, the gear ratio is 3:2, and the rear wheel goes around three times for every two times the pedals go around. Since the rate the pedals go around is the same as before, the speed is half of the speed in the first part of the ride, or approximately 14.9 km/h. At this rate, it takes approximately $4/14.9 = 0.268$ hours, or 16 min to ride 4 km. In the last part of the ride Magda uses a gear ratio of 2:1, travels at $80 \cdot 60 \cdot 2 \cdot 0.66\pi/1000 \approx 19.9$ km/h, and covers 3.5 km in $3.5/19.9 = 0.176$ hours, or 11 min approximately. The entire trip takes approximately $4 + 16 + 11 = 31$ minutes.

16. We want x to be 500 and y to be 0 at the time the shell hits the target, and we use this to determine the other quantities. We want $0° < A < 90°$.
(a) Suppose Colin aims the cannon correctly and hits the target at time $t = T$, then $x(T) = 500$ and $y(T) = 0$. This means (1) $85T \cos A = 500$, and (2) $85T \sin A - 4.9T^2 = T(85 \sin A - 4.9T) = 0$. The last equation implies that $T = 0$ (which is impossible since (1) then cannot be satified), or $T = \frac{85}{4.9} \sin A$. Substituting this expression for T into (1) gives $\frac{85^2}{4.9} \sin A \cos A = 500$. Using the sine's double angle identity, we get $\frac{85^2}{9.8} \sin 2A = 500$, therefore $\sin 2A = 9.8(500)/85^2$ and $2A \approx 42.7°$ or $180° - 42.7° =$

$137.3°$, thus $A \approx 21.4°$ or $68.6°$.

(b) The graph of t vs. y is a parabola, and we want to find its vertex, whose t-coordinate is midway between the two t-axis intercepts. If $A \approx 21.4°$ then the two t-axis intercepts are 0 and $T = \frac{85}{4.9}\sin A \approx 6.32$, so the highest point occurs at $T/2 \approx 3.16$ seconds, when the altitude is $y(T/2) \approx 48.9$ metres. On the other hand, if $A \approx 68.6°$ then $T \approx 16.16$ and the highest point occurs at $T/2 \approx 8.08$ seconds when the altitude is $y(T/2) \approx 319.8$ metres.

(c) If the target is 5 km away, then $\sin 2A = 9.8(5000)/85^2$ has no solutions A. The physical reason is that the target is too far away for the shell to reach.

17. We find the total amount each company will pay over 5 years. Alexor will pay \$70 000 per year $\times 5$ years, or \$350 000 over a five-year time period. Bantek will pay \$65 000 the first year, \$65 000(1.04) in year two, $\$65\,000(1.04)(1.04) = \$65\,000(1.04)^2$ in year three, $\$65\,000(1.04)^3$ in year four, and $\$65\,000(1.04)^4$ in year five, for a total number of dollars over a five-year period of $65\,000 + 65\,000(1.04) + 65\,000(1.04)^2 + 65\,000(1.04)^3 + 65\,000(1.04)^4 = \$352\,060.97$ This is a (finite) geometric series of the form $S_n = a + ar + ar^2 + \cdots + ar^{n-1}$ with $a = 65\,000$, $r = 1.04$, and $n = 5$. Recall that S_n can also be expressed as $S_n = \frac{a(r^n-1)}{r-1}$. It may be worth computing S_5 both ways. At Calico you will be paid $\$60\,000/4 = \$15\,000$ for the first quarter-year, \$15 000(1.02) for the second-quarter year, $\$15\,000(1.02)^2$ for the third-quarter year, etc. up to twenty quarter-years. Using the formula above for S_n with $a = 15\,000$, $r = 1.02$ and $n = 20$ we get \$364 460.55, so the largest total amount paid over five years would be from Calico. (Computing S_{20} directly is also possible, but lengthy.)

18. We find areas after each step for a few steps, then detect a pattern to find the area of the limiting figure. The area of the initial equilateral triangle with side ℓ is $\frac{\sqrt{3}}{4}\ell^2$. In Step 1, each of the 3 sides sprouts an equilateral triangle with side $\frac{\ell}{3}$. Replacing ℓ by $\frac{\ell}{3}$ in the formula above for the area of the original triangle gives the area of one of the 3 smaller triangles, $\frac{\sqrt{3}}{4} \cdot \frac{\ell^2}{9}$. Therefore the total area after Step 1 is

$$\frac{\sqrt{3}\,\ell^2}{4} + 3 \cdot \frac{\sqrt{3}}{4} \cdot \frac{\ell^2}{9}.$$

Notice that each of the original 3 sides has been replaced with 4 sides, giving a figure with $4 \cdot 3 = 12$ sides after Step 1. In Step 2, each of these $4 \cdot 3$ sides sprouts an equilateral triangle with side $\frac{\ell}{3} \cdot \frac{1}{3}$ and area $\frac{\sqrt{3}}{4} \cdot \frac{\ell^2}{9} \cdot \frac{1}{9}$, so the total area after Step 2 is

$$\frac{\sqrt{3}\,\ell^2}{4} + 3 \cdot \frac{\sqrt{3}}{4} \cdot \frac{\ell^2}{9} + 4 \cdot 3 \cdot \frac{\sqrt{3}}{4} \cdot \frac{\ell^2}{9} \cdot \frac{1}{9}.$$

Each of the $4 \cdot 3$ sides after Step 1 has been replaced with 4 sides, giving a figure with $4 \cdot (4 \cdot 3) = 48$ sides. After each step we add 4 times the number of triangles that were added in the previous step, and the areas of the new triangles are $\frac{1}{9}$ times the area of one of the triangles added in the previous Step, so the total area after Step 3 is

$$\frac{\sqrt{3}\,\ell^2}{4} + 3 \cdot \frac{\sqrt{3}}{4} \cdot \frac{\ell^2}{9} + 4 \cdot 3 \cdot \frac{\sqrt{3}}{4} \cdot \frac{\ell^2}{9} \cdot \frac{1}{9} + 4 \cdot 4 \cdot 3 \cdot \frac{\sqrt{3}}{4} \cdot \frac{\ell^2}{9} \cdot \frac{1}{9} \cdot \frac{1}{9}.$$

Notice that we add a term that is $\frac{4}{9}$ times the last term in the previous sum. In the limit as the number of steps goes to infinity we get the area of the Koch snowflake as an infinite geometric series

$$\frac{\sqrt{3}\,\ell^2}{4} + \frac{\sqrt{3}\,\ell^2}{12} + \frac{\sqrt{3}\,\ell^2}{12} \cdot \frac{4}{9} + \frac{\sqrt{3}\,\ell^2}{12} \cdot \left(\frac{4}{9}\right)^2 + \cdots + \frac{\sqrt{3}\,\ell^2}{12} \cdot \left(\frac{4}{9}\right)^{n-1} + \cdots$$

whose sum is

$$\frac{\sqrt{3}\,\ell^2}{4} + \frac{\sqrt{3}\,\ell^2}{12} \cdot \frac{1}{1 - \frac{4}{9}}$$

or $2\sqrt{3}\ell^2/5$.