UBC Grade 11/12 Solutions 2000

1. Let the edge lengths be a, b, and c. We know their products in pairs. With suitable labelling $bc = 720$, $ca = 1000$, $ab = 1250$. Multiply. We get $(abc)^2 = (720)(1000)(1250)$, and therefore $abc = 30000$.
   We could instead use elimination to find say a (then b and c are immediate.) The first calculation exploited the fact that volume is symmetric in a, b, and c.

2. The first equation can be rewritten as $(x-y)(x+y) = 0$, so it represents a "curve" made up of the familiar lines $y = x$ and $y = -x$. The second equation is the equation of the circle with center $(k, 0)$ and radius 1. Because of symmetry, we can look first at the case $k \geq 0$, and then reflect across the y-axis. A glance at a picture shows that there are 4 solutions if $k = 0$. Increase k. When the circle passes through the origin ($k = 1$) the number of solutions drops to 3. Then it jumps immediately back to four. It stays at four until the circle becomes tangent to $y = x$, when the number of solutions drops to 2. That happens when $k = \sqrt{2}$. And after that there are no solutions. So there are no solutions when $|k| > \sqrt{2}$, two solutions when $|k| = \sqrt{2}$, four when $1 < |k| < \sqrt{2}$, three when $|k| = 1$, and four when $|k| < 1$.
   The problem can also be handled algebraically. Eliminate y^2. We obtain the equation $2x^2 - 2xk + k^2 - 1 = 0$. There are no roots when the discriminant $8 - 4k^2$ is negative, that is, when $|k| > \sqrt{2}$. When the discriminant is 0, there is one root, so two solutions. Below that, there are two roots, and hence four solutions, except in the case when one of the roots is 0, when there are only three solutions.

3. Look at the "middle" square of the tromino, and colour the inner vertex of this square red. This red dot must go to one of the 49 points where inner lines of the chessboard meet. And for each such point, the tromino can be placed in 4 ways. So there are 4×49 ways.
   Or we can think about the orientation of the tromino: it can be like an 'L', like an upside-down 'L', and so on, four possibilities in all. If it is

like 'L', the vertical part can be in any of 7 columns, and for each such column the tromino can be put in 7 different places, for a total of 49. We get identical counts for the other 3 possibilities, for a total of 196. Or we can be less efficient, and examine where the inner square S of the tromino goes. For each of the 4 corner squares of the chessboard, there is only one way of placing S. For each of the 24 remaining edge squares, there are two ways of placing the tromino so S is on that square. And for each of the remaining 36 squares of the chessboard, there are 4 ways of placing S. So there are $4 + 48 + 144$ ways.
If we are in a topological mood, we can make the board into a torus, overlapping the north and south rows of squares, also the east and west rows. This yields another argument that the answer is 4×7^2.
For an $m \times n$ chessboard, we can in the same way show that there are $4(m-1)(n-1)$ ways of placing the tromino.

4. We could use "algebra" but careful reasoning should get us through. Janet got to the top 12 minutes before Fred, so she got halfway 6 minutes before Fred. That is, Fred got to the halfway point 6 minutes after Janet passed Alicia. In the remaining 10 minutes until he caught Alicia, he covered as much territory as she had in 16 minutes. When Fred caught up to Alicia, she had been hiking for 30 minutes longer than Fred. Every 16 minutes, she "loses" 6 minutes, so when they met she had travelled 80 minutes (and Fred had travelled 50). Thus Alicia had reached the halfway point in $80 - 16$ minutes, and therefore took 128 minutes in all. She got to the top at 6:38.
How might the algebra go, if we proceed more or less mechanically? Let A, F, and J be the time, in minutes, that our hikers took. Then $F = J + 12$. Since Janet caught Alicia at the midpoint, $A/2 = J/2 + 30$. Sixteen minutes later, the fraction of the Grind Alicia had covered is $(A/2 + 16)/A$. Fred had travelled for time $A/2 + 16 - 30$, and hence
$$(A/2 + 16)/A = (A/2 - 14)/F.$$
From the first two equations $F = A - 48$. If we substitute into the third equation, we find that $A = 128$.

5. By symmetry, it doesn't matter where P is. So let it be on top of the circle of radius 2. The line PQ crosses the small circle precisely if Q lies between the points labelled A and B.

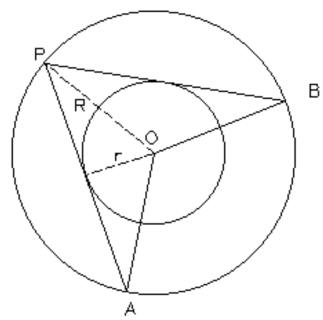

Figure 1: Two Circles

The probability of this is the ratio of $\angle AOB$ to a full rotation. There are various ways of finding the size of $\angle AOB$. Perhaps recall that (by a general result) this is twice $\angle APB$. But $\angle APO$ is easy to find, for it has sine equal to r/R where r and R are the radii of the two circles. For our particular numbers, we conclude that $\angle APO$ is a $30°$ angle, so $\angle AOB$ is a $120°$ angle, and the probability is $1/3$.

6. Please see the diagram below. The diagonal AC is a perpendicular bisector of the fold line. Let $\theta = \angle CAB$, let d be the length of the diagonal, and f the length of the fold line. Then $(f/2)/(d/2) = \tan\theta$, so $f = d\tan\theta$. By the Pythagorean Theorem, $d = 26$. And $\tan\theta = 10/24$. The fold line has length $260/24$. If we wish we can avoid mention of $\tan\theta$ and use the language of similar triangles.

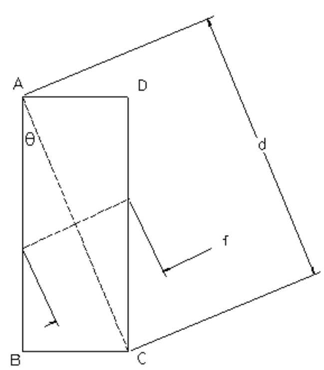

Figure 2: Folding a Rectangle

7. The region consists of three squares, of combined area 338, and four triangles. Since $5^2+12^2=13^2$, the inside triangle $\mathcal{T}$ is right-angled. So it has area 30. Observe that the other three triangles also have area 30. This is trivially true for one of them. So now look for example at the northwest triangle in the picture. Rotate it clockwise through $90°$ about its lower right-hand corner. This shows that it has the same base and height as $\mathcal{T}$, so the same area.
   Or else we could find areas as $(xy\sin\theta)/2$, and get our conclusion from the fact that complementary angles have the same sine. Now add up. We get 458.

8. There are other trigonometric identities that are mentioned sometime in grade 12 and that students might look for analogues of. Part (c) requires logarithms, which are often not done until fairly late in grade 12.
   (a) We can square x and y, then subtract. It is easier to note that $x^2-y^2=(x+y)(x-y)=a^ta^{-t}=1$.
   (b) We get

$$\sinh_a(t)=(a^{2t}-a^{-2t})/2=(a^t+a^{-t})(a^t-a^{-t})/2=2\cosh_a(t)\sinh_a(t).$$

   (c) Let $u=10^t$. The equation can be rewritten as $(u+1/u)/2=20$, that is, $u^2-40u+1=0$. Solve as usual: $u=20\pm\sqrt{399}$. It follows that $t=\log(20\pm\sqrt{399})$. For people who like decimals, the results are roughly ± 1.6017883.

9. Join the top and bottom of the kite. We have divided the kite into two equal triangles. We will decide on the best angle between "a" and "b." Think of the triangle as having base b. Then the larger the "height," the larger the area. It is clear that we reach greatest height by having the stick of length a perpendicular to the base. The maximum possible area of the kite is ab.

10. (a) It is probably best to start by saying that we use 5 \$2 coins, or 4, or 3, or 2, or 1, or 0. If we use 5 \$2 coins, the game is over, so that gives 1 way of making change. If we use 4 \$2 coins, we need to produce 2 dollars with loonies and quarters. That can be done in 3 ways (2 loonies, or 1, or 0.) If we use 3 \$2 coins, the remaining \$4 can be done in 5 ways, and so on. So the number of ways of making change

is $1+3+5+7+9+11$, that is, 36.

(b) This is handled in the same way. For the number of ways of changing a \$20, we end up having to find $1+3+5+\cdots+19+21$, which is 121. For the number of ways of changing a \$50, we are looking at the sum $1+3+\cdots+49+51$. For this it may be worthwhile to use develop some machinery. We get $(26)^2$.

There are more geometrical ways of proceeding. For example, the sum $1+3+\cdots+11$ can be seen to be 6^2 from the dot pattern below. In

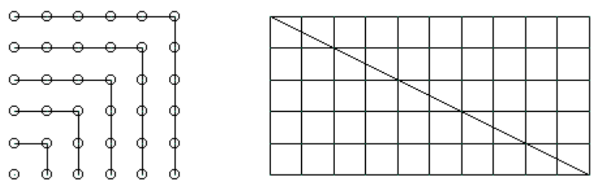

Figure 3: Making Change

general, suppose that we are trying to make change for a $2k$-dollar bill. Represent the use of x two-dollar coins and y one-dollar coins (which determines the number of quarters) by the point (x, y). We want to count the number of pairs (x, y) such that $2x + y \le 2k$. These are the points with integer coordinates in a certain triangle. One can count them directly, or work instead with a rectangle.

11. Let the legs be a and b. Then $a^2 + b^2 = 225$ amd $ab = 32$. So $a^2 + 2ab + b^2 = 289$, and $a + b = 17$. The perimeter is 32. There are harder (and less symmetrical) ways to solve the problem.

12. The triangle that sticks out is similar to $\triangle ABC$. Because it has area equal to 0.64 times the area of $\triangle ABC$, its sides are 0.8 times the sides of $\triangle ABC$. Unfold, and label things as in the diagram below. The line PQ is equidistant from RS and BC. It follows that $\triangle APQ$ has 0.9 times the linear dimensions of $\triangle ABC$, and so in particular the fold PQ has length 21.6. There are more complicated ways to solve the problem.

13. We could calculate. There is a lot of symmetry, and we can take the vertices to be $P = (0, 1)$, $(\sqrt{3}/2, \pm 1/2)$, $(1/2, \pm\sqrt{3}/2)$, and so on, so the calculation is not even very long. But one can prove a stronger

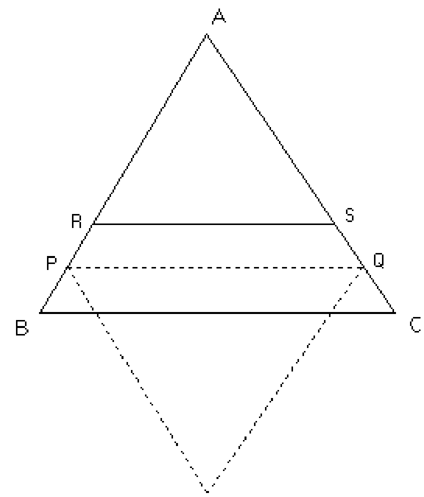

Figure 4: The Folded Triangle

result in an easier way.

Let P be *any* point on the circle. We will work with a regular $2n$-sided polygon. Then the vertices can be divided into diametrically opposite pairs. Take any such pair A, B. Then $\angle APB$ is a right angle. So by the Pythagorean Theorem $(PA)^2 + (PB)^2 = (AB)^2 = 4r^2$. (The argument is not quite right, P could be one of A or B, but then again $(PA)^2 + (PB)^2 = 4r^2$.) So we get a contribution of $4r^2$ from each of the n pairs, for a total of $4nr^2$. For the 12-gon the sum is $24r^2$.

14. The general term has shape $9 + 23n$ where n is a non-negative integer. We want $9+23n$ to be a perfect square, say $9+23n = x^2$, or equivalently $(x-3)(x+3) = 23n$. Since 23 is prime, it must divide one of $x-3$ or $x+3$. The cheapest way to achieve this with $x > 3$ is to make $x+3 = 23$. The next cheapest is to make $x-3 = 23$. And the next cheapest after that is to make $x+3 = 46$. So the perfect squares are 20^2, 26^2, and 43^2.

15. Look at the entire region $\mathcal{R}$ covered by the diagram. We can think of $\mathcal{R}$ as the semicircle on "4", plus the semicircle on "3," plus the triangle. We can also think of $\mathcal{R}$ as the semicircle on "5" plus the shaded region. But the two small semicircles have combined area equal to that of the

big semicircle. This can be done by direct computation, but it is also a consequence of a natural generalization of the Pythagorean Theorem. It follows (by cancellation) that the shaded region has the same area as the triangle, namely 6.

Comment. Hippocrates of Chios (around -450) did essentially the same thing with the right-angled triangle with equal legs, concluding that each "lune" has area $1/4$. Thus a natural region with curvy sides can have a nice area. There is reason to think that Hippocrates was interested in the problem of "squaring the circle" with compass and straightedge.

16. There are 125 such numbers, so adding them up is unappealing. *Imagine* listing all the numbers and adding. There are 25 numbers with last digit 1, 25 with last digit 3, and so on. So the sum of the last digits is $25(1+3+\ldots+9)$, that is, 625. Similarly, sum of the "tens" digits is 625, as is the sum of the "hundreds" digits. So the sum is $625+625(10)+625(100)$, that is, 69375.

17. The given curve has an oval shape. Imagine drawing circles with center the origin and radius r, starting with small r and letting r increase. As r grows, after a while the circle fails to meet the curve. The last r for which it does meet the curve represents the largest distance a point on the curve can be from the origin. If we substitute r^2-y^2 for x^2 in the equation of the curve, we obtain

$$17y^4-2r^2y^2+r^4-16=0.$$

This has a (real) solution as long as the discriminant $64(17-r^4)$ is non-negative. So the largest solution is reached when $r=17^{1/4}$.

Comment. Consider the curve $x^a+2^ay^a=2^a$ for various values of a. It turns out that if $a\le 2$, then the largest distance from the origin is 2, but that if $a>2$ the largest distance is greater than 2. The "intuitively obvious" fact that in the ellipse $x^2+4y^2=4$ the point furthest from the origin is at a distance 2 from the origin is therefore perhaps not so obvious.

18. If we try to use the calculator in a brute force way, we run out of digits. We will use a trick, well, not really a trick, more like a Method. Let

$$N=(100+\sqrt{10001})^3+(100-\sqrt{10001})^3.$$

It turns out that N is an integer. This can be done for example by thinking about expanding each of the cubes, and noticing that the terms that involve $\sqrt{10001}$ cancel.
We rewrite things as

$$(100+\sqrt{10001})^3 = N + (\sqrt{10001}-100)^3.$$

Thus the quantity that we want to calculate is an integer plus $(\sqrt{10001}-100)^3$. The calculator will now work. The first two non-zero digits are 12.

19. In order for N to be such an integer, we need to have

$$a+0.321 \leq \sqrt{N} < a+0.322$$

for some integer a. Equivalently, we want

$$a^2+(0.642)a+(0.321)^2 \leq N < a^2+(0.644)a+(0.322)^2.$$

This is pretty easy to arrange: we need to make sure that there is an integer between the lower bound and the upper bound. That will already be true if we pick $a=500$. Then the left bound is about 250321.1, the right bound is about 250322.1, so we can take $N=250124$.
A closely related way of looking at things is to note that the difference $\sqrt{n+1}-\sqrt{n}$ between consecutive squares is equal to $1/(\sqrt{n+1}+\sqrt{n})$. By the time we reach 250000, such differences are (barely) less than 0.001. At 250000 (and therefore 250001) the first three digits after the decimal point are 000. At 251000, they are 999. So we go in order from .000 to .999.
We can get away much more cheaply. We can do a brute force seach, which (for the first three digits after the decimal point) should be feasible with almost any programmable calculator. Or we can use more sophisticated tools, such as continued fractions, to find suitable a. It turns out that the smallest N that works is 2054.