UBC Grade 8/9 Solutions 2000

1. (a) The 10 kg of dry are 88% "mushroom powder" and 12% water, so we have 8.8 kg of powder. This represents 10% of the fresh weight, which is therefore 88 kg.
(b) Note from part (a) that 10 kg of dry come from 88 of fresh, so 11 of fresh yield 10/8 of dry.

2. There are 18 small triangles, 8 midsized ones, and 2 big. We can make the problem almost twice as easy by taking advantage of symmetry. The right angle of a triangle either points northwest or southeast, and there are 9 small northwest pointing triangles, 4 midsized ones, and 1 big. The counting is not much harder on the analogous 4×4 board. We count the northwest facing triangles of various sizes. There are 16 small ones, 9 of the next size, 4 of the next, and 1 of the next. Interesting numbers! In general with an $n \times n$ board there is a total of $2(1^2 + 2^2 + \cdots + n^2)$ triangles of all sizes.

3. We can use "rates" machinery, but that is overkill. Slice each lamb into 6 portions (for this the lamb could be a long rectangle.) In an hour L eats 3 portions, W eats 2, and F eats 1, altogether 1 lamb. So the time taken is 3, L eats 1.5 lambs, W eats 1, and F eats 0.5.

4. The volume of a disk of diameter d and fixed thickness is kd^2 for some constant k. It so happens that here $k = (0.1)(\pi/4)$, but that doesn't matter. The combined volumes are $k(6^2 + 6^2 + 7^2)$, that is, $k(11^2)$, the volume of a disk of diameter 11.

5. Look at the "middle" square of the tromino, and colour the inner vertex of this square red. The red dot must go to one of the 49 points where inner lines of the chessboard meet. And for each such point, the tromino can be placed in 4 ways. So there are 4×49 ways.
Or we can think about the orientation of the tromino: it can be like an 'L', like an upside-down 'L', and so on, four possibilities in all. If it is

like ‘L’, the vertical part can be in any of 7 columns, and for each such column the tromino can be put in 7 different places, for a total of 49. We get identical counts for the other 3 possibilities, for a total of 196.
Or we can be less efficient, and examine where the inner square S of the tromino goes. For each of the 4 corner squares of the chessboard, there is only one way of placing S. For each of the 24 remaining edge squares, there are two ways of placing the tromino so S is on that square. And for each of the remaining 36 squares of the chessboard, there are 4 ways of placing S. So there are $4 + 48 + 144$ ways.
If we are in a topological mood, we can make the board into a torus, overlapping the north and south rows of squares, also the east and west rows. This yields another argument that the answer is 4×7^2.
For an $m \times n$ chessboard, we can in the same way show that there are $4(m-1)(n-1)$ ways of placing the tromino.

6. There are general techniques available (Chinese Remainder Theorem) but they only become necessary with much larger numbers. The number is a multiple of 11, so we list them all, 11, 22, 33, 44, 55, 66, 77, 88, 99, 110, 121, 132, 143, 154, 165, 176, 187, 198, 209. If we group in fives, there are 3 left, so our number is 3 more than a multiple of 5, and therefore ends in 3 or 8. That leaves the candidates 33, 88, 143, 198. And only 143 leaves a remainder of 3 on division by 4.
It is a little faster to note that since we have a remainder of 3 on division by 4 and also on division by 5, it is clear (?) that there is a remainder of 3 on division by 20. So we are looking for a multiple of 11 among 23, 43, 63, 83, and so on.

7. Note that 20 gallons “cost” 135 seconds, while a second costs 20/75 dollars. So one gallon costs 135/75 dollars, that is, \$1.80.

8. There are many possible approaches. A fairly simple one is to fill out the octagon to a square by adding four little triangles at the corners. The combined area of these four little triangles is easy to find: they can be placed together to make a 1×1 square.
For the side of the square we need some machinery. The little triangles have hypotenuse 1, so by the Pythagorean Theorem they have legs $1/\sqrt{2}$. (Or four have combined area 1, so two have combined area 1/2, and the legs are $1/\sqrt{2}$.) Thus the big square has side $1 + \sqrt{2}$, hence area $3 + 2\sqrt{2}$. The area of the octagonis $2 + 2\sqrt{2}$, about 4.828.

9. In 40 seconds, Anna does a fraction $40/72$ of the track, so Boris does $32/72$, that is, four ninths of the track. To do one ninth takes 10 seconds, to do nine ninths takes 90.

10. Some "algebra" is probably unavoidable. Let the edge lengths be a, b, and c. We know their products in pairs. With suitable labelling $bc = 720$, $ca = 1000$, $ab = 1250$. Multiply. We get $(abc)^2 = (720)(1000)(1250)$, and therefore $abc = 30000$.
We could instead use elimination to find say a (then b and c are immediate.) The first calculation exploited the fact that volume is symmetric in a, b, and c.

11. Put five marks on the perimeter of the top of the cake, equally spaced around the perimeter, and slice, with the slices meeting at the center. It is clear that everyone gets equal amounts of side frosting. We show that everyone gets equal amounts of top frosting (and therefore cake.) Any slice is either a triangle of base $32/5$ and height 4, or (if it goes around a corner) is made up of two triangles of combined base $32/5$ and height 4. Thus all slices have equal top area. The same idea can be used to slice a rectangular cake "fairly" into any number of pieces.

12. Join the top and bottom of the kite. We have divided the kite into two equal triangles. We will decide on the best angle between "a" and "b." Think of the triangle as having base b. Then the larger the "height," the larger the area. It is clear that we reach greatest height by having the stick of length a perpendicular to the base. The maximum possible area of the kite is ab.

13. (a) It is probably best to start by saying that we use 5 \$2 coins, or 4, or 3, or 2, or 1, or 0. If we use 5 \$2 coins, the game is over, so that gives 1 way of making change. If we use 4 \$2 coins, we need to produce 2 dollars with loonies and quarters. That can be done in 3 ways (2 loonies, or 1, or 0.) If we use 3 \$2 coins, the remaining \$4 can be done in 5 ways, and so on. So the number of ways of making change is $1 + 3 + 5 + 7 + 9 + 11$, that is, 36.
(b) This is handled in the same way. For the number of ways of changing a \$20, we end up having to find $1 + 3 + 5 + \cdots + 19 + 21$, which is 121. For the number of ways of changing a \$50, we are looking at the sum $1 + 3 + \cdots + 49 + 51$. For this it may be worthwhile to develop

some machinery. We get $(26)^2$.

There are more geometrical ways of proceeding. For example, the sum $1 + 3 + \cdots + 11$ can be seen to be 6^2 from the dot pattern below.

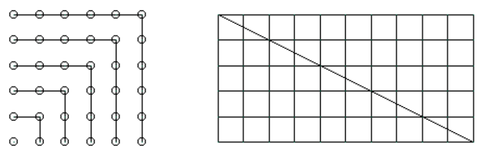

Figure 1: Making Change

In general, suppose that we are trying to make change for a $2k$-dollar bill. Represent the use of x two-dollar coins and y one-dollar coins (which determines the number of quarters) by the point (x, y). We want to count the number of pairs (x, y) such that $2x + y \leq 2k$. These are the points with integer coordinates in a certain triangle. One can count them directly, or work instead with a rectangle.

14. She did first, second, and third in 42 minutes and 36 seconds. We ask how long altogether the first, second, and fourth quarter took. Clearly 1 minute 30 seconds less, that is, 41:06. Add 42:36 and 41:06. That represents the time for two first halves plus a second half, and is 83:42. But a first half takes as much as a second half, so three halves take 83:42, one half takes 27:54, two take 55:48.

It is more pleasant, specially with a calculator, to use pure decimal notation. And "algebra" can replace thinking. Let x be say the third quarter time. Then the fourth quarter time is $x - 1.5$. So the second half (and hence the first half) took $2x - 1.5$, so the 3/4 mark was reached in $3x - 1.5$. Thus $3x - 1.5 = 42.6$. It follows that $x = 14.7$ and the total time is 55.8.

The problem also yields to a modified "guess and check." The fourth quarter is faster than the third. The average over three quarters is a bit more than 14 minutes a quarter, so maybe the fourth quarter took 14:00. Then the third would be 15:30, so the first half would be 29:30, the first three-quarters 45:00, quite a bit too much. If the fourth quarter took say 13:00, we find in the same way that the first three-quarters takes 42:00. It looks as if changing the fourth quarter time by a certain amount changes the three-quarters time by three times as much. Maybe therefore the fourth quarter time should be 13:12. That gives 14:42 for the third, 27:54 for half, 42:36 for three-quarters (good!), and 55:48 for the whole.

15. We don't need to know the volume of a cone, in fact that makes the solution more complicated. The larger cone of water is just the smaller cone scaled up by a linear factor of $6/4$. So the volume of the larger cone is the volume of the smaller, multiplied by $(6/4)^3$. The volume is 54.

16. We could use "algebra" but reasoning should get us through. Janet got to the top 12 minutes before Fred, so she got halfway 6 minutes before Fred. That is, Fred got to the halfway point 6 minutes after Janet passed Alicia. In the remaining 10 minutes until he caught Alicia, he covered as much territory as she had in 16 minutes. When Fred caught up to Alicia, she had been hiking for 30 minutes longer than Fred. Every 16 minutes, she "loses" 6 minutes, so when they met she had travelled 80 minutes (and Fred had travelled 50.) Thus Alicia had reached the halfway point in $80 - 16$ minutes, and therefore took 128 minutes in all. She got to the top at 6:38.
How might the algebra go, if we proceed more or less mechanically? Let A, F, and J be the time, in minutes, that our hikers took. Then $F = J+12$. Since Janet caught Alicia at the midpoint, $A/2 = J/2+30$. Sixteen minutes later, the fraction of the Grind Alicia had covered is $(A/2 + 16)/A$. Fred had travelled for time $A/2 + 16 - 30$, and hence

$$(A/2 + 16)/A = (A/2 - 14)/F.$$

From the first two equations $F = A - 48$. If we substitute into the third equation, we find that $A = 128$.

17. There are 125 such numbers; adding them up is unappealing. *Imagine* listing all the numbers and adding. There are 25 numbers with last digit 1, 25 with last digit 3, and so on. So the sum of the last digits is $25(1 + 3 + \ldots + 9)$, that is, 625. Similarly, sum of the "tens" digits is 625, as is the sum of the "hundreds" digits. So the desired sum is $625 + 625(10) + 625(100)$, that is, 69375.