UBC Grade 7 Solutions 2000

1. With Lease-a-Loser we start out 3 dollars behind, but gain 5 cents a mile, that is, a dollar every 20 miles. Past 60 miles Lease-a-Loser gives the better deal.

2. There are 18 small triangles, 8 midsized ones, and 2 big. We can make the problem almost twice as easy by taking advantage of symmetry. The right angle of a triangle either points northwest or southeast, and there are 9 small northwest pointing triangles, 4 midsized ones, and 1 big.
   To someone who is quick to find the answer, one can suggest looking at the analogous 4×4 board and counting the northwest facing triangles of various sizes. There are 16 small ones, 9 of the next size, 4 of the next, and 1 of the next. Interesting numbers! In general with an $n \times n$ board there will be a total of $2(1^2 + 2^2 + \cdots + n^2)$ triangles of all sizes.

3. We can work backwards. If we know how much he left a store with, to find what he entered with we add 1 then double, or double and add 2. So he entered store 4 with \$5, store 3 with \$12, store 2 with \$26, and store 1 with \$54.
   Or let x be the amount he enters store 1 with. Then he leaves with $x/2 - 1$, leaves store 2 with $(x/2 - 1)/2 - 1$, and so on. Finally, using an unpleasant number of brackets, we get

$$(((x/2 - 1)/2 - 1)/2 - 1)/2 - 1 = 1.50.$$

   The simplest way to solve the equation is to unwind it, bracket by bracket, which brings us back to the first solution.

4. (a) The 10 kg of dry are 88% "mushroom powder" and 12% water, so we have 8.8 kg of powder. That represents 10% of the fresh weight, which is therefore 88 kg.
   (b) Note from part (a) that 10 kg of dry come from 88 of fresh, so 11 of fresh yield 10/8 of dry.

5. Work first with Sophie. Colour truth days blue, lie days red. Today has to have a different colour than yesterday. (We may want to put the days in a circle.) Thus today can only be Wednesday, Friday, Saturday, or Sunday. By the same argument with Mary, today is Monday, Tuesday, Thursday, or Saturday. Thus today is Saturday. We could have looked at the days one at a time instead of in bulk. Less pleasant!

6. We could use "rates" machinery, but that is overkill. Slice each lamb into 6 portions (for this the lamb could be a long rectangle.) In an hour L eats 3 portions, W eats 2, and F eats 1, altogether 1 lamb. So the time taken is 3, L eats 1.5 lambs, W eats 1, and F eats 0.5.

7. Some will recognize that the binary expansion of 650 tells the story. But we can just list the prizes 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024. It is reasonably clear that 512 was one of the prizes, leaving 138, which comes from 128 +8 +2. Four questions (#1, #3, #7, #9) were answered correctly.

8. Look at the "middle" square of the tromino, and colour the inner vertex of this square red. This red dot must go to one of the 49 points where inner lines of the chessboard meet. And for each such point, the tromino can be placed in 4 ways. So there are 4×49 ways.
   Or we can think about the orientation of the tromino: it can be like an 'L', like an upside-down 'L', and so on, four possibilities in all. If it is like 'L', the vertical part can be in any of 7 columns, and for each such column the tromino can be put in 7 different places, for a total of 49. We get identical counts for the other 3 possibilities, for a total of 196.
   Or we can be less efficient, and examine where the inner square S of the tromino goes. For each of the 4 corner squares of the chessboard, there is only one way of placing S. For each of the 24 remaining edge squares, there are two ways of placing the tromino so S is on that square. And for each of the remaining 36 squares of the chessboard, there are 4 ways of placing S. So there are $4 + 48 + 144$ ways.
   If we are in a topological mood, we can make the board into a torus, overlapping the north and south rows of squares, also the east and west rows. This yields another argument that the answer is 4×7^2.
   To someone who found the 8×8 problem easy, one can suggest the generalization to $m \times n$ chessboards. The same arguments show that there are $4(m-1)(n-1)$ ways of placing the tromino.

9. We count. Between 12:00 and 2:00, also between 10:00 and 12:00, there is at least one 1 showing in the hour part. That takes care of 4 hours. Look at the remaining 8 hours. In each hour, there are 10 minutes (between :10 and :20) plus 5 minutes (:01 to :02, :21 to :22, and so on) in which there is a 1 showing, so 1 shows 1/4 of the time. That takes care of two more hours, for a total of 6, one-half of the time.

10. Imagine that toppings come in "scoops," so that a mushroom bacon and onion pizza gets three scoops. The total number of scoops is $12 + 27 + 28 + 29 + 30$, that is, 126. Each pizza has 3 scoops, so there are 126/3 pizzas. Of these, $12 + 27$ have meat, and therefore 3 are vegetarian.

11. In 40 seconds, Anna does a fraction 40/72 of the track, so Boris does 32/72, that is, 4 ninths of the track. To do 1 ninth takes 10 seconds, to do 9 ninths takes 90.

12. Call the region of interest $\mathcal{R}$. Let P' be the point on the left circle that is directly above P, and Q' the point on the right circle directly above Q. The rectangle $PQQ'P'$ has area 2. We show that $\mathcal{R}$ also has area 2.
    The rectangle consists of half of circle $\mathcal{C}$, together with two small roughly triangular bits. These two bits can be cut off and reassembled to make the part of $\mathcal{C}$ that lies below $P'Q'$. So $\mathcal{C}$ consists of two roughly triangular bits plus a semicircle, as does rectangle $PQQ'P'$, and therefore their areas are the same.

13. We calculate, at least for a while. The remainders are 1, 1, 2, 3, 1, 0, 1, 1, 2, 3, 1, 0, 1, and so on. There is reason to believe that there is cycling, with period 6. If that is so, the 39-th remainder is then same as the third remainder, namely 2. Note that the "next" remainder always depends only on the preceding two remainders. So once we see a consecutive pair that we have seen before, there is indeed cycling. It is not difficult to argue that if we replace 4 by any positive integer n, in the long run there must be cycling. But already when $n = 5$ it takes a while.

14. There are general techniques available (Chinese Remainder Theorem) but they only become necessary with much larger numbers. Our number is a multiple of 11, so we list them all, 11, 22, 33, 44, 55, 66, 77,

88, 99, 110, 121, 132, 143, 154, 165, 176, 187, 198, 209. If we group in fives, there are 3 left, so our number is 3 more than a multiple of 5, and therefore ends in 3 or 8. That leaves 33, 88, 143, 198 as candidates. And only 143 leaves a remainder of 3 on division by 4.

It would be a little faster to note that since we have a remainder of 3 on division by 4 and also on division by 5, it is clear (?) that there is a remainder of 3 on division by 20. So we are looking for a multiple of 11 among 23, 43, 63, 83, and so on.

15. She did first, second, and third in 42 minutes and 36 seconds. How long altogether did the first, second, and fourth quarter take? Clearly 1 minute 30 seconds less, that is, 41:06. Add 42:36 and 41:06. That represents the time for two first halves plus a second half, and is 83:42. But a first half takes as much as a second half, so three halves take 83:42, one half takes 27:54, two take 55:48.

    It is more pleasant, especially with a calculator, to use pure decimal notation. And "algebra" can replace thinking. Let x be say the third quarter time. Then the fourth quarter time is $x - 1.5$. So the second half (and hence the first half) took $2x - 1.5$, and therefore the 3/4 mark was reached in $3x - 1.5$. Thus $3x - 1.5 = 42.6$. This gives $x = 14.7$, and therefore the total time is 55.8.

    The problem also yields to a modified "guess and check." The fourth quarter is faster than the third. The average over three quarters is a bit more than 14 minutes a quarter, so maybe the fourth quarter took 14:00. Then the third would be 15:30, so the first half would be 29:30, the first three-quarters 45:00, quite a bit too much. If the fourth quarter took say 13:00, we find in the same way that the first three-quarters takes 42:00. It looks as if changing the fourth quarter time by a certain amount changes the three-quarters time by three times as much. Maybe therefore the fourth quarter time should be 13:12. That gives 14:42 for the third, 27:54 for half, 42:36 for three-quarters (good!), and 55:48 for the whole.

16. (a) Listing is perfectly feasible. It should be in some sense systematic, so as to provide insurance against making a mistake. We get 16 positive divisors. A more advanced way of seeing this is to list the primes 2, 3, 5, 7. Then we make a divisor of 210 by writing y (yes) or n (no) under the primes, to indicate whether the prime is "in" or "out." More generally,

if we want to count the number of positive divisors of say $2^a \times 3^b \times 5^c \times 7^d$, we list 2, 3, 5, 7 and under 2 put 0, 1, 2, …, or a, meaning we are grabbing zero 2's, one 2, two 2's, …, a 2's, and do something similar under 3, 5, 7. The number of divisors is $(a+1)(b+1)(c+1)(d+1)$.
(b) We do not need the general formula above to count the number of divisors of 420. The divisors of 420 are the ones not divisible by 2 (so they are the divisors of 105, 8 of them), also twice the divisors of 105, also four times the divisors of 105, in all (3)(8). For 840, we also want numbers that are 8 times a divisor of 105, giving a total of (4)(8).

17. We could use "algebra" but careful reasoning should get us through. Janet got to the top 12 minutes before Fred, so she got halfway 6 minutes before Fred. That is, Fred got to the halfway point 6 minutes after Janet passed Alicia. In the remaining 10 minutes until he caught Alicia, he covered as much territory as she had in 16 minutes. When Fred caught up to Alicia, she had been hiking for 30 minutes longer than Fred. Every 16 minutes, she "loses" 6 minutes, so when they met she had travelled 80 minutes (and Fred had travelled 50.) Thus Alicia had reached the halfway point in $80 - 16$ minutes, and therefore took 128 minutes in all. She got to the top at 6:38.