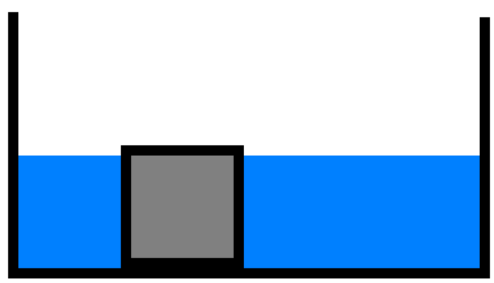
UBC Grade 11/12 Problems 1999

1. A 600 pound pumpkin was entered in a contest. When it arrived, it was 99% water. The pumpkin sat for days in the hot sun, lost some weight (water only), and is now 98.5% water. How much does it weigh?

2. A tub has a flat bottom with area 5000 square cm, and vertical sides. There is water in the tub to a depth of 20 cm.
(a) A concrete cube with sides 25 cm is placed on the floor of the tub. How much does the water level rise?
(b) A second identical cube is placed on the floor of the tub. How much further does the water level rise now?



3. Find 50 consecutive odd numbers which add up to 10000.

4. The polynomial $(x-1)(x-2)$ divides $ax^4 + bx^3 + 1$. Find a and b.

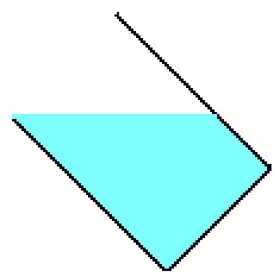
5. For what values of c does the equation $|x-3| + |x-33| = c$ have
(i) no solutions;
(ii) two solutions;
(iii) more than two solutions?

6. Find the maximum value reached by $3x + 4y$ as the point (x, y) roams over the circle $x^2 + y^2 = 1$. Hint: Draw the line $3x + 4y = c$ for a few values of c.

7. Compute, correct to three significant figures, the roots of the equation

$$(1.5 \times 10^{-4})x^2 - 2475x + (3.2 \times 10^{-4}) = 0.$$

8. A cylindrical glass has inner radius 4 cm and inner height 12 cm, and is full of water.
(a) Through what angle (to the nearest degree) should we tilt the glass so that 70 cubic cm of water flow out?
(b) What about if we start with water level 1 cm below the top?

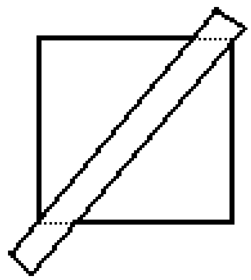


9. There are 52 cards in a standard deck, including 12 *face cards* (four Jacks, four Queens, and four Kings). A deck of cards is thoroughly shuffled, and three cards are dealt. What is the probability that among these cards there is at least one face card?

10. Let $f(\theta) = \cos\theta + \sin\theta$.
(a) How many solutions does the equation $f(\theta) = 5/4$ have in the interval $0 < \theta < \pi/2$?
(b) What is the largest possible value of $f(\theta)$?
(c) If $f(\theta) = 5/4$, find the values of $\tan\theta$.

11. If a convenience store charges \$1.50 per sixteen ounce cup, it can sell 960 cups a week of its Burpee carbonated drink. For each 1 cent reduction in the price, it can sell an additional 24 cups per week. The ingredients for sixteen ounces of Burpee cost the store 2 cents. The cup itself costs the store an additional 4 cents. What price should the store charge for a cup of Burpee in order to make its weekly profit as large as possible?

12. A 3 cm wide ruler is placed on a 15 cm by 15 cm piece of paper, with one edge of the ruler passing through a corner of the paper, and the *other* edge of the ruler passing through the diagonally opposite corner. What proportion of the paper is covered by the ruler?



13. (a) How many digits are there in the decimal expansion of 2^{1999}?
(b) Find the leftmost digit.

14. Observe that $\dfrac{1}{1-u} + \dfrac{1}{1+u} = \dfrac{2}{1-u^2}$.
(a) For $x \neq \pm 1$, simplify the expression

$$\frac{1}{1+x} + \frac{2}{1+x^2} + \frac{2^2}{1+x^4} + \frac{2^3}{1+x^8} + \frac{2^4}{1+x^{16}}.$$

(b) Find, correct to 1999 decimal places,

$$\frac{1}{1+3} + \frac{2}{1+3^2} + \frac{2^2}{1+3^4} + \frac{2^3}{1+3^8} + \frac{2^4}{1+3^{16}} + \cdots + \frac{2^{50}}{1+3^{2^{50}}}.$$

15. The sides of a triangle are in geometric progression, and the smallest side is equal to 1. What are the possible values of the largest side?

16. (a) Find two numbers a, b such that $a^3 + b^3 = 1$ and $3ab = -3$.
(b) Verify the identity

$$(u+v)^3 - 3uv(u+v) - (u^3+v^3) = 0.$$

(c) Find an expression for the real solution of $x^3 + 3x - 1 = 0$. Why is there only one real solution?