# UBC Grade 11/12 Solutions 1999

**1.** 99% of 600 is 594, so the pumpkin had 594 lbs of water originally. Let the weight of the water lost be $x$, in pounds. Then, after sitting in the sun, the pumpkin has lost $x$ lbs of water, and therefore $x$ lbs of total weight. We also know that the pumpkin is now 98.5% water. Therefore, $\frac{594-x}{600-x} = 0.985$ Solving this, we see that $x = 200$ lbs, and therefore the total weight of the pumpkin is $600 - x = 400$ lbs.

**2.** (a) We must be careful. There are two cases: the block can either have so much volume that it displaces the water so that it is totally submerged, or it can have less volume, and thus has a part sticking up out of the water line. Assume that for one block, that the cube does not end up being totally submerged. Let the height of the amount submerged be $h$, in cm. Then, the amount of water displaced by the block is $25^2h$ and this is equal to the volume formed by the waterline and the tub minus the original volume of water, $5000 \times h - 5000 \times 20$. Therefore, $4375 \times h = 100000$ and $h = 160/7$. Notice that this is less than the side of the cube (25 cm), so our assumption was correct. The question asks for how much the water level rose. This is $160/7 - 20 = 20/7$, or about 2.857 cm.
(b) Again, assume that the blocks are not totally submerged. Then, the change in volume is $2 \times 25^2 \times h$ and this is equal to $5000^2 \times h - 50000 \times 20$. This time, $h$ is larger than 25 cm, so the blocks are submerged completely. Therefore, we must recalculate with the assumption that the blocks are completely under water. Let the depth of the new water level be $l$. Then, $l$ must be equal to the sum of the volumes of the original water mass and the new concrete masses, divided by the base area of the tub. $l = (100000 + 2 \times 25^3)/5000 = 105/4$. How much did the water level rise? $105/4 - 20 = 25/4$. The question asks how much *more* the water rises after adding the second block... this is $25/4 - 22/7 = 97/28$, or about 3.393 cm.

**3.** The numbers are 151, 153, ..., 249. Note that arithmetic progressions, geometric progressions may not yet have been done. Here is a way of solving the problem without the "formula." Guess that the numbers are $-49, -47, \ldots, -1, 1, 3, \ldots, 47, 49$. Whoops, the sum is zero, bad guess. Fix it up by adding 10000/50 to each number.
Or (equivalently) note that 25 pairs with pairwise sum 400 will do the job. This can be achieved by choosing pairs symmetrical about 200.

**4.** $a = 7/8$, $b = -15/8$ We can use "long division" by $x^2 - 3x + 2$, ending up with a couple of linear equations. Or we can let $ax^4 + bx^3 + 1 = (x-1)(x-2)Q(x)$, and put $x = 1$, repeat with $x = 2$, getting again a couple of linear equations.
Or note that a polynomial $P(x)$ is divisible by $(x-1)(x-2)$ iff it is divisible by $x - 1$ and by $x - 2$, and recall that $P(x)$ is divisible by $x - a$ iff $P(a) = 0$.

**5.** (i) $c < 30$;
(ii) $c > 30$;
(iii) $c = 30$ The point of the question is to visualize and draw $y = |x-3| + |x-33|$. I would want to remind students that $|a-b|$ is the distance between $a$ and $b$. As $x$ moves from 3 to 33, the sum of the distances of $x$ to 3 and 33 is a steady 30. For each step that we take to the right from 33, the sum of the distances increases by 2. And for each step to the left from 3, .... The sketch flows out automatically.
The symmetry is worth remarking on. One can even move the $y$-axis, or the curve, so as look instead at $|x - (-15)| + |x - 15|$.
Formal calculation ($|u| = u$ when $u \geq 0$, $-u$ when $u < 0$) is also possible.
There are other geometric ways of proceeding. For example, rewrite the equation as $|x - 3| = -|x - 33| + c$. Sketch the curves $y = |x - 3|$, and $y = -|x - 33|$ (well-known to students). "Lift" $y = |x - 33|$ until it meets $y = |x - 3|$.

**6.** 5
Calculus works OK, specially if we maximize $3\cos\theta + 4\sin\theta$. But many of the students have not done calculus, and anyway there are other attractive approaches, as there are for most of the standard "max–min" word problems.
The hint is intended to bring out a more geometric solution, and an

idea which is useful elsewhere. By thinking about lines $3x + 4y = c$, students can see that if $c$ is to be as big as possible we must have tangency. And by looking at slopes they can find that the point of tangency is $(3/5, 4/5)$.
Here is a solution from the hindsight is 20–20 department. Use the following norm identity (which goes back to Diophantus):

$$(ax + by)^2 + (bx - ay)^2 = (a^2 + b^2)(x^2 + y^2).$$

In particular, if $x^2 + y^2 = 1$ then $(3x + 4y)^2 = 25 - (4x - 3y)^2$, and the maximum value of $(3x + 4y)^2$ is 25.

**7.** The roots, to 3 significant figures, are $1.65 \times 10^7$ and $1.29 \times 10^{-7}$. Most (all?) calculators give 0 for the small root, if we use the standard "quadratic formula" in the standard way—close in one sense, but hopelessly incorrect in another. Grade 12 students should be able to understand what went wrong.
Somewhat beside the point here, but of interest to students, is the fact that calculators keep "guard digits." This can be illustrated by having the calculator find $\sqrt{2}$, or $\pi$, and subtracting the *displayed value* of the result. An amusing fact: my cheap (under \$10) TI displays only 8 digits, while my expensive (\$16) Casio displays 10, but they work to the same internal accuracy. TI seems to be still run by engineers, not marketing people!
The simplest way to find the small root is to use the fact that $q$ is the product of the roots of $x^2 + px + q = 0$. Or else we can divide $ax^2 + bx + c = 0$ through by $x^2$, let $y = 1/x$, obtaining the equation $cy^2 + by + a = 0$. Equivalently, the usual formula can be transformed, by rationalizing the numerator, to $\dfrac{2c}{-b \mp \sqrt{b^2 - 4ac}}$, which works nicely for the small root (and chokes on the big one.) Or else we can look at the more symmetric form $ax + b + c/x$.
The "Solve" button on some calculators will find both solutions.

**8.** The height of the glass, after a certain point, is irrelevant.
(a) If $\theta$ is the angle of tilt, then the "air space" is half of a cylinder of radius 4 and height $8\tan\theta$. So $\tan\theta = \dfrac{70}{64\pi}$, and $\theta$ is about 19 degrees.
(b) Add $16\pi$, filling the glass to the brim, then pour it out, plus 70 more cc. We get $\tan\theta = \dfrac{16\pi + 70}{64\pi}$, so $\theta$ is about 31 degrees.

**9.** $1 - \dfrac{40 \cdot 39 \cdot 38}{52 \cdot 51 \cdot 50}$, that is, 47/85, roughly 0.553. There are many other ways of counting. The answer is probably larger than intuition would suggest.

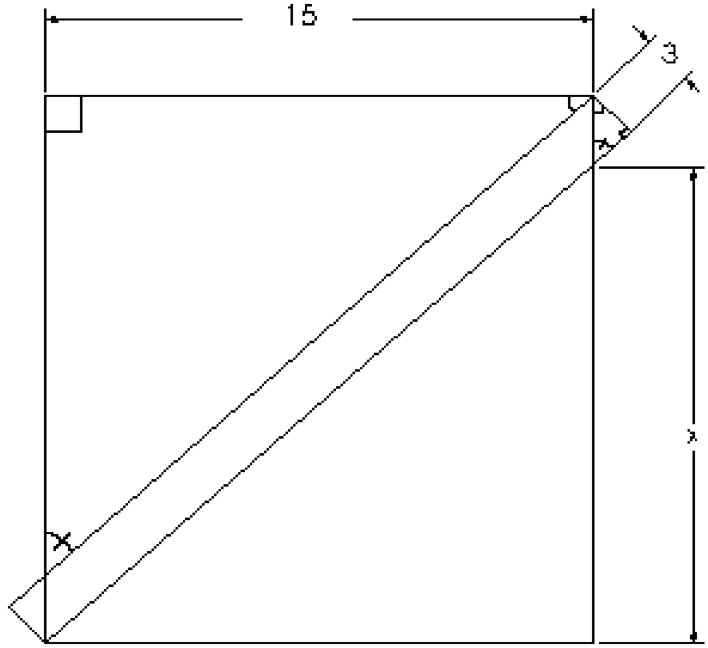
**10.** (a) two;
(b) $\sqrt{2}$;
(c) $\tan\theta = (16 \pm 5\sqrt{7})/9$, about 0.3079 and 3.2476
For parts (a) and (b), one can be very geometric. Draw $y = \cos\theta$, and also $y = -\sin\theta$. The distance between the two curves is $f(\theta)$. Any reasonably good picture "shows"—if students are willing to believe that cos and sin look like they have been told they look—that $f(\theta)$ climbs until $\theta = \pi/4$, then falls. So the maximum is $\sqrt{2}$. (Note also that part (b) is a simpler variant of the problem of maximizing $3x + 4y$ subject to $x^2 + y^2 = 1$.)
A more manipulational approach, more or less necessary for part (c), is to write

$$(\cos\theta + \sin\theta)^2 = 1 + 2\sin\theta\cos\theta = 1 + \sin 2\theta.$$

Thus the maximum value of $(\cos\theta + \sin\theta)^2$ is reached when $\sin 2\theta = 1$. Or else use $(\cos\theta + \sin\theta)^2 = 2 - (\cos\theta - \sin\theta)^2$. For finding $\tan\theta$ when $f(\theta) = 5/4$, we can use the calculator. Since $\sin 2\theta = 9/16$, $2\theta$ is roughly 34.229 or 145.701 degrees, and now we compute $\tan\theta$.
A fast way to an *exact* expression for (c) is to note that $\sin^2\theta + \cos^2\theta = 1$ and $\sin\theta\cos\theta = 9/32$. Divide. So $\tan\theta + 1/\tan\theta = 32/9$, and we get a quadratic equation for $\tan\theta$.

**11.** 98 cents.
Students can be asked: can you find a formula for the profit $P(x)$ if the store takes $x$ cents off the \$1.50 price? Now can you graph it? Grade 12 students know that the maximum of a downward facing parabola is halfway between the roots.
We could instead look directly at $P(x+1) - P(x)$. By dropping the price by 1 cent, the store gains an additional $24(144 - x - 1)$ but loses $960 + 24x$, for a net "gain" of $24(103 - 2x)$. So it should drop up to and including $x = 51$.

**12.** The marked triangles are similar. Thus, $\dfrac{3}{15} = \dfrac{15-x}{\sqrt{225+x^2}}$. This leads to a quadratic equation with solution $x = (125 \pm 35)/8$. The positive root must be rejected, because it is larger than the piece of paper. Thus, $x = 90/8$, and the area uncovered is $2 \times \left(\frac{1}{2}(15)\frac{90}{8}\right)$. The total area is $15^2$, and therefore, the percentage covered is $\frac{15^2 - 15\times\frac{90}{8}}{15^2} = 25\%$.




**13.** (a) $1999 \times \log(2) = 601.759$. Thus, $2^{1999}$ has 601 powers of 10. In other words, there are 602 digits, as the number of digits in $10^n$, where $n$ is an integer, is $n+1$ for $n \geq 1$.
(b) the first few digits are 574065.
Direct computation presumably fails with any standard calculator. One general way to proceed is to use logarithms (to the base 10). If $N = 2^{1999}$, we have $\log N = 1999 \log 2$. So the calculator gives that $\log N$ is roughly 601.7589613. That answers part (a). For part (b), note that $N$ is roughly equal to $10^{601} \times 10^{0.7589613}$.
There are ways of avoiding, or seeming to avoid, logarithms. Direct calculation failed because of overflow. One way (among many)

to avoid overflow is to note that $2^{10} = 1.024 \times 10^3$, and therefore $2^{2000} = (1.024)^{200} \times 10^{600}$. Now the calculator gives that $N$ is roughly equal to $57.4065 \times 10^{600}$. There are many variants of this idea.
We need to be a little cautious about using logarithms to solve this question since at the time of the workshop students may not yet have worked with logarithms.

**14.** (a) $\dfrac{2^5}{1-x^{32}} - \dfrac{1}{1-x}$ if $x \neq \pm 1$ (adding $\dfrac{1}{1-x}$ brings about collapse);
(b) The results from part (a) imply that the sum is $\frac{2^{51}}{1-3^{2^{51}}} - \frac{1}{1-3}$. The result is thus $0.5000\ldots$ There are about $10^{15}$ more zeros (about a billion books full of zeros) before the next non-zero digit.

**15.** Let the sides be 1, $r$, and $r^2$. Since $r \geq 1$, a necessary and sufficient condition is $r^2 < 1+r$, since the longest side of a triangle cannot exceed the sum of the other two sides. We now have a quadratic inequality in $r$, solving, we find that $1 \leq r < (1+\sqrt{5})/2$. The longest side is $r^2$, or $(3+\sqrt{5})/2$. Therefore, valid solutions are $1 \leq m < (3+\sqrt{5})/2$, where $m$ is the longest side of the triangle.

**16.** (a) $b = -1/a, a^3 - 1/a^3 = 1$, substituting $x = a^3$, we get a quadratic $x^2 - x - 1 = 0$, which is readily solved. $a = \sqrt[3]{(1+\sqrt{5})/2}$, $b = -1/a$ (or vice-versa);
(c) Let $x = u + v$. Then, the we can substitute into the expression in part (b), and compare with the equation $x^3 + 3x - 1 = 0$. From the coefficients of the linear terms, we see that $-3uv = 3$, and from the constant terms, we see that $u^3 + v^3 = 1$. This is exactly the system in part (a), and rearranging terms, we find that $x = u+v = a-1/a$, where $a$ is as above. The uniqueness will not seem obvious, since grade 12 students may not notice that since $x^3$ and $3x$ are increasing, so is $x^3 + 3x - 1$.
It can be pointed out that this is a particular case of the Cardano formula for the roots of the cubic, and that the general case is basically no more difficult, sort of.