UBC Grade 11/12 Problems 1998

1. Surrey has a square-grid road system. Streets running from north to south are numbered with every integer from 120 to 196 (inclusive) in order from west to east. Adjacent streets are 100 m apart. Sets of traffic lights are located on every eight streets, starting from 120th Street (i.e., 120th St, 128th St, etc.). Kay intends to drive along 88th Avenue from 120th Street to 196th Street at 60 km/h.
(a) How long will it take Kay if all traffic lights are green?
(b) Suppose traffic lights alternate green and red for one minute intervals simultaneously, and Kay starts just when the light turns green at 120th Street. Now how long will it take Kay? [Assume a constant speed, i.e., ignore acceleration/deceleration.]

2. A spherical crystal ball of radius 5 cm is placed on a stand which has a 5 cm deep hole of circular cross-section of radius 4 cm (i.e., the hole is a circular cylinder). Find the gap between the bottom of the crystal ball and the bottom of the hole.

3. A square town, 50 km $\times$ 50 km, has three hospitals. One is located in the centre of the town, while the other two hospitals occupy opposite corners of the town. Assume that the population is evenly spread throughout the town. What fraction of the population lives within 25 km of a hospital?

4. During the 1998/99 hockey season Pavel scores 20 more goals and attempts 200 more shots than Mark. However Mark scores on 15% of his shots whereas Pavel scores on 12.5% of his shots. How many goals does Pavel score during the 1998/99 season?

5. Calculate $\dfrac{1}{\sqrt{2}+1} + \dfrac{1}{\sqrt{3}+\sqrt{2}} + \dfrac{1}{\sqrt{4}+\sqrt{3}} + \cdots + \dfrac{1}{\sqrt{100}+\sqrt{99}}$

6. Solve for $x : (x^2 - 5^x + 5)^x = 1$.

7. Solve for $x: 2^{2x+1} - 2^{x+4} - 2^x + 2^3 = 0$.

8. Given that $\log_{10} 2 = 0.3010$ and $\log_{10} 3 = 0.4771$, evaluate $\log_{10} 31$ to within 1% accuracy.

9. An 8.8 cm × 17.6 cm tile is being painted with a "shrinking-square" pattern. The painter first paints an 8.8 cm × 8.8 cm square, then a 4.4 cm × 4.4 cm square, then a 2.2 cm × 2.2 cm square, etc., such that the dimensions of a particular square is half the dimensions of its preceding square. The painting continues until the string of squares reaches the opposite side of the tile. What fraction of the tile is painted?

10. Find all values of $\theta, 0 \leq \theta \leq 2\pi$, for which the equation $x^2 + 2(\sin\theta)x + \sin\theta = 0$ has two distinct real roots.

11. Find all values of x for which $\dfrac{x+1}{x-1} < \dfrac{x}{x-2}$.

12. Find the sum of the series:

$$\frac{1}{1} + \frac{1}{1+2} + \frac{1}{1+2+3} + \cdots + \frac{1}{1+2+\cdots+1999}$$

13. How would you estimate the number of jelly beans in a cylindrical glass jar filled to the top which is closed by a painted cover? Assume that you are unable to lift the jar. [The closest guess wins the jar of jelly beans!]

14. Let $p_1 = 1, p_2 = 1 - \frac{1}{2!}, p_3 = 1 - \frac{1}{2!} + \frac{1}{3!}, p_{n+1} = p_n + \frac{(-1)^n}{(n+1)!}$ for $n = 3, 4, ...$ For $n > 5$, calculate p_n to two decimals of accuracy. Prove your result.