UBC Grade 11/12 Solutions 1998

1. (a) The car needs to travel 76 blocks (total length 7.6 km) at a speed of 60 km/h. Thus, the time required is $7.6/60 = \frac{19}{150}$ hours = 7.6 minutes.
(b) The car can travel 60 km/h = 1 km/min = 10 blocks in one minute. Let's make a simple time-line.

Block #	0	8	10	16	24	26	32
Light	G	G	R	Wait for G	G	R	Wait for G
+ time			+1 min	+1 min		+1 min	+1 min

At block 16, the car has reached a red traffic light. We add 1 minute of time here because the light will turn green 1 minute after the car reaches the 10th block regardless of how fast the car travels. The same reasoning applies at the 32nd block. The blocks from 32 to 64 are similar to the case for 0 to 32 because we are starting the car from a green light. Thus, the transit time for the first 64 blocks is 8 minutes. The final 12 blocks don't require the car to make a stop for a red light, and we can calculate the elapsed time by 1.2 km/60 km/h = 1.2 minutes. Summing, we get a total transit time of (1.2 + 8) = 9.2 minutes.

2. There is a 3-4-5 right triangle within the cross-section of the globe (diagram on next page). Therefore, 2 cm of the globe is below the top of the stand, and there are 3 cm between the bottom of the globe and the bottom of the hole.

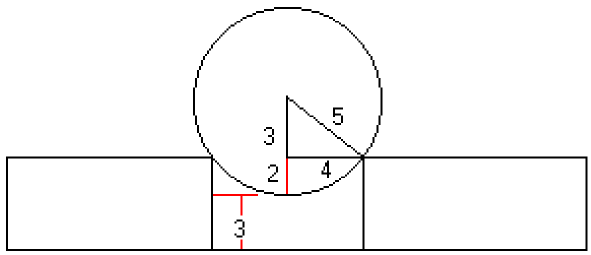

Figure 1: Diagram for Question 2

3. The corner circles meet the centre one at the tangent points along the square. The area not within 25 km of a hospital is half that of the area not covered by the hospital in the middle. (Diagram on next page.)

$$\tfrac{1}{2}(A_{square} - A_{circle}) = \tfrac{1}{2}(50^2 - 25^2\pi)$$

The fraction not within 25 km is the previous area over the total area,

$$\frac{50^2 - 25^2\pi}{2(50^2)} = \frac{1}{2} - \frac{\pi}{8}$$

Thus, the fraction of the population within 25 km is $1 - \left(\frac{1}{2} - \frac{\pi}{8}\right) = \frac{1}{2} + \frac{\pi}{8}$.

4. Let P stand for Pavel, M stand for Mark, g stand for goals and s stand for shots. Then, we are given the equations:
(1) $P_g = M_g+20$, (2) $P_s = M_s+200$, (3) $P_g = 0.125P_s$, (4) $M_g = 0.15M_s$
Thus, $P_g = 0.15M_s + 20 = 0.15(P_s - 200) + 20 = 0.15P_s - 30 + 20 = 0.15P_s - 10$; and, from Eq. (3), $P_s = P_g/0.125$. Substituting for P_s,we get $P_g = 0.15P_g/0.125 - 10$, implying that $P_g = 50$.

5. Multiply each term by the conjugate of the denominator on the top and the bottom. The expression simplifies to:

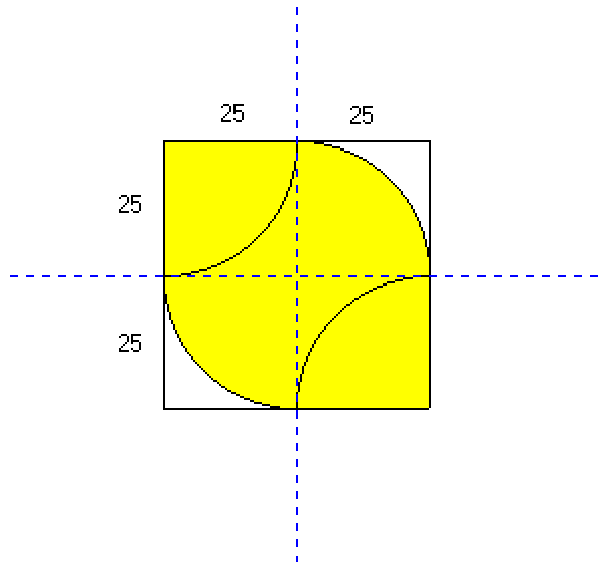

Figure 2: Diagram for Question 3

$$\frac{\sqrt{2}-\sqrt{1}}{2-1}+\frac{\sqrt{3}-\sqrt{2}}{3-2}+\frac{\sqrt{4}-\sqrt{3}}{4-3}+\cdots+\frac{\sqrt{100}-\sqrt{99}}{100-99}$$

Each term's denominator is one, and the sum of the numerators is a telescoping sum. Cancelling, we get $10-1=9$.

6. Three cases can arise:
I) The exponent is zero, and the base is not zero. This corresponds to $x=0$.
II) The base is one, the exponent is any real number. Solving $x^2-5x+5=1$, we find $x=1,4$.
III) The base is -1, the exponent is even. Solving $x^2-5x+5=-1$, we find $x=2,3$. $x=2$ works here, but we must discard $x=3$ because it isn't even.
Thus, $x=0,1,2,4$.

7. Let $y=2^x$. Then we can write the equation as $2(2^{2x})-2^4(2^x)-2^x+2^3=2y^2-16y-y+8=0$. Thus, $2y(y-8)-1(y-8)=(2y-1)(y-8)=0$, and $y=\frac{1}{2},8$. Since $y=2^x$, x must thus be -1 or 3.

8. $32 = 2^5$, so $\log 32 = 5 \log 2 = 1.505$. $30 = 3 \times 10$, so $\log 30 = 1 + \log 3 = 1.4771$. Use these two results to construct a linear approximation to the log function for $30 \le x \le 32$. We average the values to find an approximation for the $\log 31 : (1.505 + 1.4771)/2 = 1.49105$.

9. The area of the squares is in an infinite geometric progression, $8.8^2 + 4.4^2 + 2.2^2 + 1.1^2 + \cdots$, with initial value 8.8^2 and ratio $(\frac{1}{2})^2$. Thus, the area of the squares in total is $8.8^2 \left(\frac{1}{1-\frac{1}{4}}\right) = \frac{4}{3} \times 8.8^2$. The total area of the tile is $8.8 \times 17.6 = 2 \times 8.8^2$ and the fraction painted is thus $(\frac{4}{3} \times 8.8^2)/(2 \times 8.8^2) = 2/3$.

10. Check the discriminant of the quadratic equation. We need $4\sin^2\theta - 4\sin\theta > 0$ or $\sin^2\theta - \sin\theta > 0$. Then, $(\sin\theta)(\sin\theta - 1) > 0$, which is positive for $\sin\theta < 0$ or $\sin\theta > 1$. Since the sine function never goes beyond 1, we must have $\sin\theta < 0$, which implies that $\pi < \theta < 2\pi$.

11. Note that the two functions have vertical asymptotes at $x = 1$ and $x = 2$. Plotting the two functions, we see that $\dfrac{x+1}{x-1}$ is less than $\dfrac{x}{x-2}$ in the regions $x < 1$ or $x > 2$.

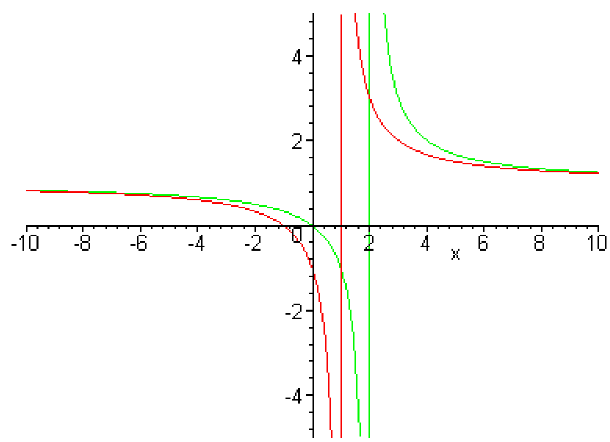

Figure 3: Graph for Question 11

12. Recall that the sum of the first k integers is $\frac{1}{2}k(k+1)$. Observe the identity:

$$\frac{2}{k(k+1)} - \frac{1}{k(k+1)} = \frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$$

Each term in the sum is the inverse of a sum of the first k integers. Let S be the desired sum.

$$S = \sum_{k=1}^{1999} \frac{2}{k(k+1)}, \text{ so } \frac{S}{2} = \sum_{k=1}^{1999} \frac{1}{k(k+1)} = \sum_{k=1}^{1999} (\frac{1}{k} - \frac{1}{k+1})$$

And thus, $\frac{S}{2} = (\frac{1}{1} - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + \cdots + (\frac{1}{1999} - \frac{1}{2000})$. The telescoping sum simplifies, $\frac{S}{2} = 1 - \frac{1}{2000}$, and $S = 1.999$.

13. Let N be the number of jelly beans counted around a perimeter (circumference) and let n be a count of layers. Then the number of jelly beans in the jar is approximately $n\left(\frac{N}{2}\right)\left(1 + \frac{N}{2\pi}\right)$.

14. For $n > 5$, $p_n - p_5 = -\frac{1}{6!} + \frac{1}{7!} - \frac{1}{8!} + \frac{1}{9!} - \frac{1}{10!} + \cdots + \frac{(-1)^n}{(n+1)!}$. Therefore,

$$|p_n - p_5| = |-1| \times \left|\tfrac{1}{6!} - \tfrac{1}{7!} + \tfrac{1}{8!} - \tfrac{1}{9!} + \tfrac{1}{10!} + \cdots - \tfrac{(-1)^n}{(n+1)!}\right|$$
$$= (\tfrac{1}{6!} - \tfrac{1}{7!}) + (\tfrac{1}{8!} - \tfrac{1}{9!}) + (\tfrac{1}{10!} - \tfrac{1}{11!}) + \cdots$$
$$= \tfrac{1}{6!} - (\tfrac{1}{7!} - \tfrac{1}{8!}) - (\tfrac{1}{9!} - \tfrac{1}{10!}) - \cdots \text{ each parenthesized term positive.}$$

Thus, $|p_n - p_5| + \text{some positive terms} = \frac{1}{6!}$, implying that $|p_n - p_5| < \frac{1}{6!}$. Since $\frac{1}{6!}$ is about 0.00138, and p_5 is about 0.633, we know that $p_n = 0.633 \pm 0.0014$, which is within two decimals of accuracy.