UBC Grade 11/12 Solutions 1997

1. Put each parenthesized term over a common denominator:

$$\frac{2-1}{2} \times \frac{3-1}{3} \times \frac{4-1}{4} \times \cdots \times \frac{1996-1}{1996} \times \frac{1997-1}{1997}$$

Each of the numerators and denominators cancel, except for the first numerator and the last denominator. The answer is 1/1997.

2. Square both sides to get $x^2 = 6 + \sqrt{6 + \sqrt{6 + \sqrt{6 + \cdots}}}$
x also appears on the right side of the equation. Thus, $x^2 - x - 6 = 0$. Solve for $x : x = \frac{1}{2}(1 \pm \sqrt{25}) = 3, -2$. We must reject the negative root because the original equation implies that x is positive. Thus, $x = 3$.

3. Note that $27 = 3^3$, so the price must be divisible by 9. Let the smudged digits be represented by x and y. Then, $x.5y$ must be divisible by 9. The sum of the digits is divisible by 9 iff the number is divisible by 9. Therefore, $x+y+5$ is either of 9 or 18 – higher numbers aren't possible because x and y are *single*-digit numbers.
If $x + y = 4$, then we can have $(x, y) = (0, 4), (1, 3), (2, 2), (3, 1), (4, 0)$.
Checking each of these, we see that 27 divides \$0.54 and \$3.51.
Checking the rest of the cases ($x + y = 13$), we see that \$0.54, \$3.51, \$4.59, and \$7.56 are divisible by 27, corresponding to an individual cherry tomato plant cost of \$0.02, \$0.13, \$0.17, and \$0.28.

4. Let the distance from A to B be 1 unit. Then, the speed going downriver is 1/3, and the speed going upriver is 1/4. But, the speed downriver is also equal to the speed rowing (v_r) plus the water flow speed (v_f), and the speed upriver is equal to the speed rowing minus the water flow speed. Therefore, we have the equations:

$$\begin{aligned} 1/3 &= v_r + v_f \\ 1/4 &= v_r - v_f \end{aligned}$$

So, $2v_f = 1/12$, or $1/v_f = 24$. This means that it takes 24 hours for the piece of wood to drift from A to B.

5. If the same inflation rate applies for the next 11 months, then the total inflation over the year would be $(1.003)^{12} = 1.03659998$. Thus, the annual inflation rate is about 3.66%.

6. From the diagram, we can use trigonometry to find the other two angles. They are about 20° and 70°.

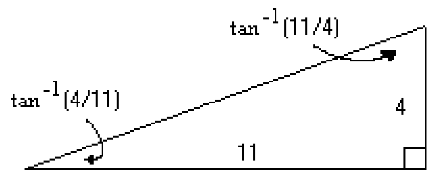

Figure 1: Trigonometry for Construction Workers

7. Pretend that we have a lot of tables piled next to each other, so that the ball can travel as shown. Then, we know that the ball will enter a corner pocket when the vertical distance travelled is a multiple of 6 and 10, since the pool table is 10 by 6. This happens at 30 feets' worth of vertical displacement. This corresponds to 3 isosceles right triangles with side length 10 and hypotenuse $10\sqrt{2}$. Since the ball travels along the hypotenuses, it has travelled $3(10\sqrt{2}) = 30\sqrt{2}$ feet.

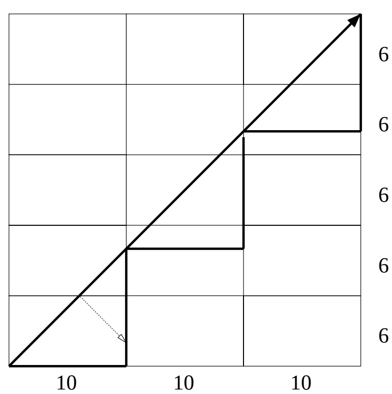

Figure 2: A Virtual Pool Table

8. Inscribing the hexagon within a circle makes it clear that each vertex of the hexagon has an angle of 120 degrees. Connecting a pair of alternate vertices thus creates an isosceles triangle with angles of 120, 30, and 30 degrees. It is clear that the desired triangle is equilateral, with side length $S = 2(4\cos 30°)$ and height $S\sin 60°$. The area is $\left(8\left(\frac{\sqrt{3}}{2}\right)\right)^2\left(\frac{\sqrt{3}}{2}\right)\times\frac{1}{2} = 12\sqrt{3}\,\mathrm{cm}^2$.

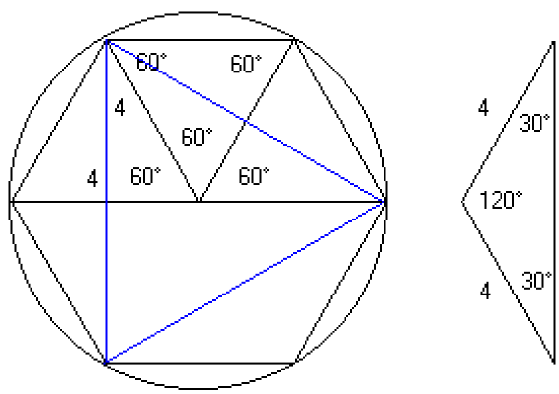

Figure 3: Triangles within a Hexagon

9. We can draw a partial cross-section of the garbage can as follows:

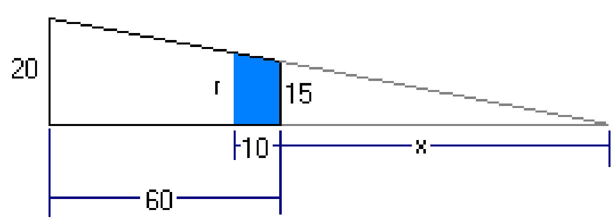

Figure 4: Cross Section of Garbage Can

Before we can utilise the given volume equation, we must first find x and r by looking at similar triangles. $\frac{15}{x} = \frac{20}{60+x}$, or $900 + 15x = 20x$. Therefore, $x = 180$. $\frac{r}{x+10} = \frac{15}{x}$, or $r/190 = 15/180$. Therefore,

$r = 95/6$. We can now calculate the amount of rainwater in the can as the difference of volumes of two cones: $(\frac{\pi}{3})(\frac{95}{6})^2(190) - (\frac{\pi}{3})(15)^2(180) =$ 7468.55...
We are not quite done yet – the question asks for how much rain fell, not how much rain there is in the garbage can. Since the garbage can has a top radius of 20, rain that fell on an area of $20^2\pi$ ended up in the can. The amount of rain that fell can be calculated as the volume of rain in the can divided by the area over which it fell: 5.943287... ≈ 5.94cm.

10. (a) It is clear that a right triangle with a 30 degree angle and a side of length 1 is a 30-60-90 triangle. Thus, the distance along the ground from the house to the launch pad is $\sqrt{3}$.
(b) We can see that one of the angles in the triangle added on after the rocket passes above the house is 120 degrees. We are also given that the side opposite this angle is 2 km. We also know that another side length is 1 km; we have enough information to use the Sine Law to calculate angle ϕ. $\dfrac{\sin(120°)}{2} = \dfrac{\sin(\phi)}{1}$; therefore, $\phi = \sin^{-1}(\sin(120°)/2)$. The desired angle is labelled θ, and is equal to $90° - (180° - 120° - \phi) =$ $55.6589...° \approx 56°$.

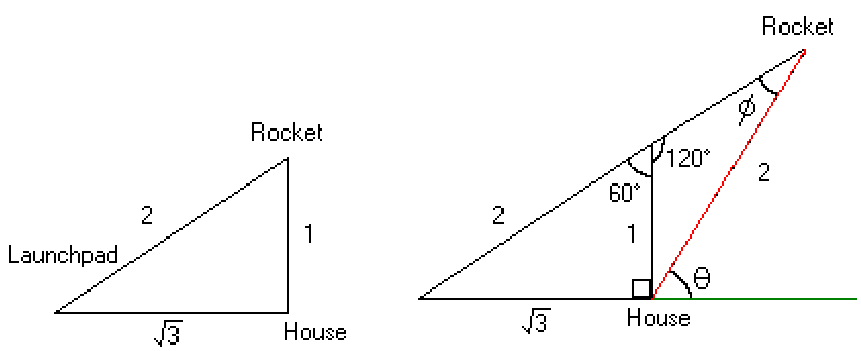

Figure 5: Trigonometry for the Rocket Question

11. See the graphs on the next page. We can solve the equivalent system $|2x/3 - y| \leq 4, |2x/3 + y| \leq 4$. Think of these equations as describing the vertical distance from a point to either the line $2x/3 - y = 0$ or the line $2x/3 + y = 0$. From the first equation, we see that we need the vertical distance from a point to the line 2x/3 - y = 0 to be less than or equal to 4. Thus, by shifting the line $2x/3 - y = 0$ up or down by 4 units, we will have found the region of the real plane in which the first inequality is satisfied. Now, the second inequality deals with the vertical distance to the line $2x/3 + y = 0$. Again, this distance needs to be at most 4. We can accomplish this by setting the boundaries as the graph $y = -2x/3$ shifted up or down by 4 units. The resulting graph is the mirror image of the first. We can plot the boundaries on the same graph to get a clearer picture of what the region we need to find looks like.

It is clear from the graph that the region in question is made up of four congruent triangles. (We can go through some algebra to find the vertices, but we'll just read them off of the graph to make things go more quickly.) The area of the half above the x-axis is that of a triangle. The base is 12 units, and the height is, of course, 4. Therefore, the total area encompassed by the original inequalities is: $2(1/2)(12)(4) = 48$ square units.

12. (a) Try completing the square:

$$\begin{aligned}
&2x^2 + 3y^2 - 4x - 5y + 1 \\
&= 2(x^2 - 2x + 1) - 2 + 3(y^2 - \tfrac{5}{3}y + \tfrac{25}{36}) - \tfrac{25}{12} + 1 \\
&= 2(x-1)^2 + 3(y - \tfrac{5}{6})^2 - \tfrac{37}{12}
\end{aligned}$$

The squared terms can be minimised by choosing a suitable (x, y) to make them zero. Thus, the minimum is $-37/12$.

(b) We can again try completing the square, but this time, we will have both x and y within one squared expression.

$$\begin{aligned}
&x^2 - xy + y^2 + y \\
&= (x^2 - xy + \tfrac{1}{4}y^2) - \tfrac{1}{4}y^2 + y^2 + y \\
&= (x - \tfrac{1}{2}y)^2 + \tfrac{1}{4}(3y^2 + 4y)
\end{aligned}$$

All that is left now is to minimise the second term (which is independent of x), because for any given y, we can pick an x to make $(x - \frac{1}{2}y)^2 = 0$.

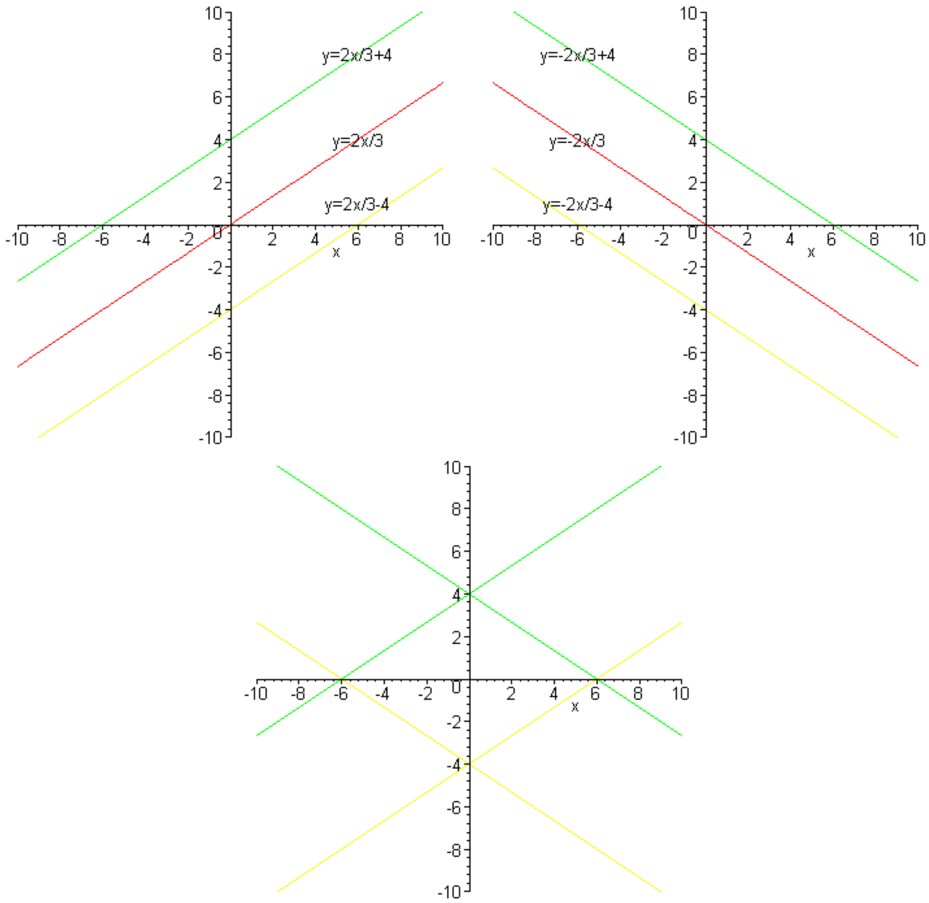

Figure 6: Graphs for Question 11

Completing the square for

$$\frac{1}{4}\left(3y^2+4y\right)=\frac{3}{4}\left(y^2+\frac{4}{3}y+\frac{4}{9}\right)-\frac{3}{4}\times\frac{4}{9}$$

We see that the minimum is $-1/3$. Thus, the minimum of the original expression is also $-1/3$.

13. (a) With just a single die, the probability that the "sum" is even or odd is obvious; they are just $1/2$. Let's look at the probabilities for the sum of two dice. There are four possible outcomes that could interest us: EE, EO, OE, OO (where the Es and Os represent the even or

oddness of the first and second die). For the cases EE and OO, the sum will be even; for the other two cases, the sum will be odd. Since the probability of getting even and the probability of getting odd on a single die is $1/2$; each of these cases occurs with probability $1/4$. Hence, the probability that the sum of two dice is even is P(EE) + P(OO) = $1/2$ (implying that the probability for an odd sum is $1 - 1/2 = 1/2$). Now, imagine adding a third die to the sum of two. We have just found that the chance that the sum of two dice being even is $1/2$, and we know that the chance that the third die is even is also $1/2$. We again have four cases: (E)E, (E)O, (O)E, (O)O, where, this time, we are lumping the first two dice together within parentheses. The chance that three dice sum to an even number is therefore P((E)E) + P((O)O) = $1/2$ again. We can continue this process for three dice and adding a fourth, grouping the first three together. Doing so, we can show that no matter how many dice we add, the probability that the sum is even is always $1/2$. Thus, the chance that the sum of 5 dice is even is also $1/2$.
(b) The product of the faces is odd if and only if all the faces are odd. This has probability $\frac{1}{2^5} = \frac{1}{32}$. Thus, the probability of the product of the faces being *even* is $1 - 1/32 = 31/32$.

14. $(n+1)^3-(n+1) = (n+1)((n+1)^2-1) = (n+1)(n)(n+2)$. This is the product of three numbers in a row, of which one must be a multiple of three, and one or two of which must be a multiple of 2. Therefore, the numbers that divide the expression for all positive n are 1, 2, 3, and 6. Substituting $n = 1$ into the expression, we find that $2^3 - 2 = 6$, so that no larger numbers will divide the expression for all positive n. If we are also counting negative integers, then we must also include $-1, -2, -3$, and -6.

15. (a) $(1-x)^3+3x-1 = 1-2x+x^2-x+2x^2-x^3-1+3x = -x^3+3x^2 = x^2(3-x) \leq 0$. Divide the real line into sections with the roots $x = 0, 3$, and let $f(x) = x^2(3-x)$.

x	$x < 0$	$x = 0$	$0 < x < 3$	$x = 3$	$x > 3$
$f(x)$	positive	0	positive	0	negative

Thus, we need $x = 0$, or $x \geq 3$.
(b) Plot the function $(1-x)^3$ and draw the line $1 - px$ such that it is

above the cubic for any $x \geq 0$. We need to have a "double root" at the

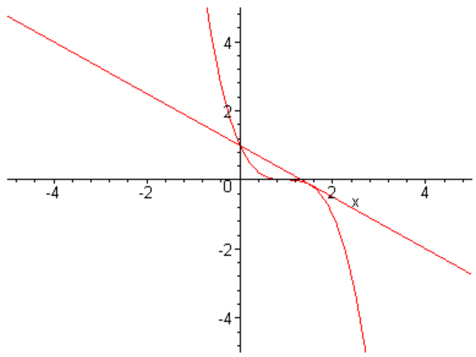

Figure 7: Graph for Question 15

second intersection in order for the line to be tangent (and thus greater than the cubic for any $x > 0$). To find this point of intersection, solve:

$$\begin{aligned}(1-x)^3 &= 1-px \\ 1-3x+3x^2-x^3+px-1 &= 0 \\ x(x^2-3x+3-p) &= 0\end{aligned}$$

For a double root to occur, the quadratic must have a double root, as we already have one root $x = 0$. From completing the square, we find that $3 - p = (\frac{3}{2})^2$, or $p = \frac{3}{4}$. Since we want the line to be above the cubic, we want the slope to be at least $\frac{-3}{4}$. $-p \geq \frac{-3}{4}$, or $p \leq \frac{3}{4}$.