UBC Grade 10 Solutions 1997

1. Let D be the distance from Vancouver to Kelowna. Then the time of flight for a round trip is $T = \dfrac{D}{240} + \dfrac{D}{360}$. The average speed for the round trip is $\dfrac{2D}{T} = \dfrac{2}{\frac{1}{240} + \frac{1}{360}} = 2(240)(360)/600 = 288\,\mathrm{kmph}$.

2. Put each parenthesized term over a common denominator. Then, we have

$$\frac{2-1}{2} \times \frac{3-1}{3} \times \cdots \times \frac{1996-1}{1996} \times \frac{1997-1}{1997}$$

Only the first numerator and the last denominator do not cancel. The answer is 1/1997.

3. Note that $27 = 3^3$, so the price must be divisible by 9. Let the smudged digits be represented by x and y. Then, $x.5y$ must be divisible by 9. The sum of the digits is divisible by 9 iff the number is divisible by 9. Therefore, $x+y+5$ is either of 9 or 18 – higher numbers aren't possible because x and y are *single*-digit numbers.
If $x + y = 4$, then we can have $(x, y) = (0, 4), (1, 3), (2, 2), (3, 1), (4, 0)$. Checking each of these, we see that 27 divides \$0.54 and \$3.51.
Checking the rest of the cases ($x + y = 13$), we see that \$0.54, \$3.51, \$4.59, and \$7.56 are divisible by 27, corresponding to an individual cherry tomato plant cost of \$0.02, \$0.13, \$0.17, and \$0.28.

4. The goals against averages that we are given are in decimal form; we must keep in mind that they could be rounded. GAAs of 2.995 to 3.005 would have been rounded to 3.00. Thus, for the first 30 games, the possible number of goals scored ranges over $[2.995 \times 30, 3.005 \times 3.005] = [89.85, 90.15]$. Since the number of goals scored must have been a whole number, we can conclude safely that 90 goals in total were scored over the first 30 games.

We are asked what the minimum number of games could have been if the GAA drops to 2.00 at the end of the season. The quickest way to do this is to let no goals in at all for the rest of the season, but we are told that there were only five shutouts, and $90/35 = 2.57$ is too big, so we need to keep adding goals. The next quickest way of dropping the GAA is to let in only 1 goal per game aside from the five shutouts. Let the number of games played after the first 30, minus the five shutouts in this latter part of the season, be x. Then, we have an equation $\dfrac{90+x}{35+x} = 2.00$. (We want only one goal per game, so we add the same number to the games and the goals.) Again, keep in mind that that 2.00 could actually be as low as 1.995 or as high as 2.005. If 2.00 "=" 1.995, then $\frac{90+x}{35+x} = 1.995$ simplifies to $x = 20.276...$ and if 2.00 "=" 2.005, then the expression simplifies to $x = 19.726...$ Both results round to $x = 20$, so, thankfully, there is no uncertainty about what the minimum number of games played was. This is $35 + 20 = 55$.

5. Let the distance from A to B be 1 unit. Then, the speed going downriver is 1/3, and the speed going upriver is 1/4. But, the speed downriver is also equal to the speed rowing (v_r) plus the water flow speed (v_f), and the speed upriver is equal to the speed rowing minus the water flow speed. Therefore, we have the equations:

$$
\begin{aligned}
1/3 &= v_r + v_f \\
1/4 &= v_r - v_f
\end{aligned}
$$

So, $2v_f = 1/12$, or $1/v_f = 24$. This means that it takes 24 hours for the piece of wood to drift from A to B.

6. Remember that there is one leap year every fourth year. 1996 was a leap year. Since $365/7 = 52 + 1/7$ (i.e. each non-leap year has 52 weeks plus a day), it follows that Jan 1 moves ahead two days in a year following a leap year and by one day in other years during the course of this century. Hence Jan 1 fell on a Monday in 1996, on a Sunday in 1995 and on a Saturday in 1994. Since $1993 - 1901 = 4 \times 23$, there were 23 leap years over that interval, and 23×3 "normal" years. Therefore, Jan 1, 1901 is $23 \times 2 + 69 = 115$ days behind Jan 1, 1993. But, $115 = 7 \times 16 + 3$; 115 days behind is equivalent to 3 days behind. Jan 1, 1993 was a Friday, we finally conclude that Jan 1, 1901 was a Tuesday.

7. All students can be put into four groups:
(i) Those who pass math and like it.
(ii) Those who don't pass math and like it.
(iii) Those who pass math and don't like it.
(iv) Those who don't pass math and don't like it.
We can devise a *Venn Diagram* to help us separate the students. Let's put all the students who like math in the L circle, and all the students who dislike it outside of the L circle. We also put the students who pass math in the P circle, while we put the students who don't pass math outside of the P circle. We are told only one thing, that is, students who pass math like it. Thus, we have something in the region where P and L overlap; put an "x" here. Note that since students who pass math like it (or so we are told), the rest of the P circle is empty. Mark this with a "0." There are two regions of the diagram we are told nothing about. Mark these with a "?."

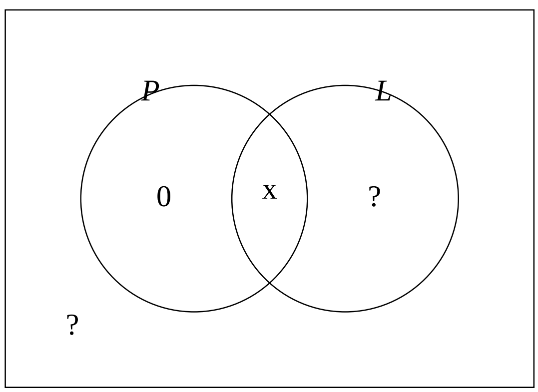

Figure 1: Venn Diagram

(a) We know nothing about the ? region within the L circle, so we cannot say with certainty that all students who like math are also within the P circle. We cannot determine the truth of (a) with certainty.
(b) The students who dislike math are outside of the L circle. For such a student to pass math, he/she must be within the P, but outside the L. Looking at our diagram, we see that this region is *empty*. Thus, (b) is true.
(c) It follows immediately from (b) that (c) is false.

(d) It follows immediately from (b) that (d) is true.
(e) This is a little tricky. It seems as if statement (e) should be true, since we have something in the region overlapped by P and L. However, we are told that students who pass math like it... We are *not* told that there exists even one such student. Thus, in actuality, the region marked "x" might actually be empty. We cannot determine the accuracy of statement (e) from the information we are given.

8. The train passes Barbara 6 km before the crossing. This corresponds to 1 hour before Barbara reaches the crossing. 1 km later, corresponding to 10 minutes, Barbara would have reached the crossing had she not been late – at this time, the train has reached the crossing. Therefore, in 10 minutes = (1/6) hours, the train travelled 6 km. Therefore, the train moves at a speed of 36 kmph.

9. Pretend that we have a lot of tables piled next to each other, so that the ball can travel as shown. Then, we know that the ball will enter a corner pocket when the vertical distance travelled is a multiple of 6 and 10, since the pool table is 10 by 6. This happens at 30 feets' worth of vertical displacement. This corresponds to 3 isosceles right triangles with side length 10 and hypotenuse $10\sqrt{2}$. Since the ball travels along the hypotenuses, it has travelled $3(10\sqrt{2}) = 30\sqrt{2}$ feet.

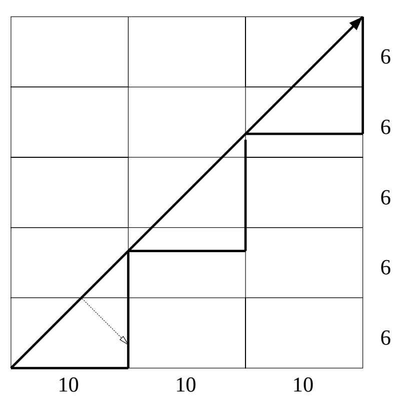

Figure 2: Virtual Pool Table

10. Let the distance walked uphill on the trip to and from Nicole's house be x. Then, the distance walked downhill is also x, because uphill on the trip to her house is downhill on the trip from. The time spent walking uphill and downhill during the entire trip is then $\frac{x}{2} + \frac{x}{6} = \frac{2x}{6}$. Since the total distance walked uphill and downhill is $2x$, the number in the denominator is David's average speed on hilly territory, 3 km/h. Note now that we can calculate his average speed overall. He travelled a total of 18 km in 6 h, so his average speed overall is 3 km/h. Since the average speed for hill and overall are equal, we must have a level ground speed that is also equal. His speed walking on level ground is 3 km/h.

We can also solve this problem algebraically. Let s be David's speed on level ground, and x be the distance walked uphill/downhill as before. Then,

$$\frac{18-2x}{s} + \frac{x}{2} + \frac{x}{6} = \frac{18-2x}{s} + \frac{2x}{3} = 6$$

That is, the time going uphill/downhill and the time walking on level ground sums to 6 hours. Solving this equation, we again find that the desired speed is 3 km/h.

Notice that this question is solvable only because of a numerical "accident." If the average speed on hills had been any other number, we would not have been able to determine the speed on level ground. Solving the previous equation for s,

$$s = \frac{18-2x}{6-\frac{2x}{k}}$$

where k is the known average speed on hilly territory, which is not equal to 3 in the case we are now considering. Since we do not know x, we only have 1 equation in 2 variables, and we cannot solve this problem. The fact that the average speed was 3 allowed us to cancel the x variable, facilitating a solution.

11. For small numbers like 2, 3, and 4, we can solve this problem by listing all possible orderings of letters, and counting the ones that have no letters in the right place. We will number our letters 1, 2, 3, 4, ..., and an ordering of these letters we will display as a string of numbers, like 3241.

For 2 letters, there are only two arrangements; one is correct (12), and the other has both letters being sent to the wrong person (21). There are $3! = 6$ ways of arranging 6 letters: (123, 132, 213, 231, 312, 321).

Two of these arrangements have no letters in the correct envelopes. The case for 4 letters begins to get tedious.

1234	1243	1324	1342	1423	1432
2134	**2143**	**2314**	2341	**2413**	2431
3124	**3142**	3214	3241	**3412**	**3421**
4123	4132	4213	4231	**4312**	**4321**

The orderings that have no numbers in the correct place are in bold. There are 9 such orderings.
The astute reader will realise that solving this type of problem by enumeration would be quite difficult if we added just one letter to the problem, as there are $5! = 120$ orderings of 5 letters. We are also asked to do seven letters; if we're really patient, we can list all the cases as before and count up the orderings that don't have any letter going to the intended recipient. There is a general formula to solve this problem for any integer n, it is

$$n! \sum_{k=0}^{n} \left(\frac{(-1)^k}{k!} \right)$$

and, applying this equation, we can find that there are 1854 bad arrangements for 7 letters and envelopes. A derivation of this formula uses the principle of "inclusion-exclusion," which is explained in various mathematical textbooks and websites. (e.g.
`http://forum.swarthmore.edu/dr.math/problems/jarrad03.27.99.html`
`http://www.unc.edu/~rowlett/Math148/notes/incl_exclude.html`
– Links last visited 2000 August 03)

12. $n^2 - n + 2 = n(n-1) + 2$. Since both terms of this expression are even, their sum is also even. The only even prime is 2; Thus, we require that $n(n-1) = 0$. In other words, $n = 0, 1$.

13. We can do a few preliminary calculations to find a pattern. $[\sqrt[3]{1}] = [1] = 1, [\sqrt[3]{2}] = [1.259...] = 1, [\sqrt[3]{3}] = [1.4422...] = 1, ..., [\sqrt[3]{8}] = [2] = 2$. It should be clear by now that $[\sqrt[3]{m}]$ is equal to the cubic root of the perfect cube that comes before m if m is not a perfect cube, or is equal to the cube root itself if m is a perfect cube. Imagine a job that pays you $\$[\sqrt[3]{m}]$ on the mth day. Then, we can interpret the question as asking on what day our average pay per day has reached \$2. That is, $[\sqrt[3]{1}] + [\sqrt[3]{2}] + \cdots + [\sqrt[3]{n}]$, our total pay up to day n, is equal to $2n$, where we view \$2 as our average pay per day.

On the first seven days, we are only making \$1 per day, since $[\sqrt[3]{m}] = 1$ for $1 \leq m < 7$ as was shown before. So far, we are seven dollars behind in total from making \$2 per day. From the eighth day to the 26th day, we are making \$2 per day, so we neither fall further behind nor gain ground on our targeted average salary. For the days from 27 to 63, we would make \$3 per day, an thus we will make up the \$7 that we lost on the first seven days, at a rate of \$1 per day. This requires seven days; so we need to work up to and including the 33rd day (there are seven days from 27 to 33). Since our average salary per day is always increasing, the solution is unique. Thus, $n = 33$.

14. (a)With just a single die, the probability that the "sum" is even or odd is obvious; they are just 1/2. Let's look at the probabilities for the sum of two dice. There are four possible outcomes that could interest us: EE, EO, OE, OO (where the Es and Os represent the even or oddness of the first and second die). For the cases EE and OO, the sum will be even; for the other two cases, the sum will be odd. Since the probability of getting even and the probability of getting odd on a single die is 1/2; each of these cases occurs with probability 1/4. Hence, the probability that the sum of two dice is even is P(EE) + P(OO) = 1/2 (implying that the probability for an odd sum is $1 - 1/2 = 1/2$). Now, imagine adding a third die to the sum of two. We have just found that the chance that the sum of two dice being even is 1/2, and we know that the chance that the third die is even is also 1/2. We again have four cases: (E)E, (E)O, (O)E, (O)O, where, this time, we are lumping the first two dice together within parentheses. The chance that three dice sum to an even number is therefore P((E)E) + P((O)O) = 1/2 again. We can continue this process for three dice and adding a fourth, grouping the first three together. Doing so, we can show that no matter how many dice we add, the probability that the sum is even is always 1/2. Thus, the chance that the sum of 5 dice is even is also 1/2.

(b) The product of the faces is odd if and only if all the faces are odd. This has probability 1/32. Thus, the probability of the product of the faces being *even* is 1 - 1/32 = 31/32.

15. $(n+1)^3-(n+1) = (n+1)((n+1)^2-1) = (n+1)(n)(n+2)$. This is the product of three numbers in a row, of which one must be a multiple of three, and one or two of which must be a multiple of 2. Therefore, the numbers that divide the expression for all positive n are 1, 2, 3, and 6. Substituting $n = 1$ into the expression, we find that $2^3 - 2 = 6$, so that no larger numbers will divide the expression for all positive n. If we are also counting negative integers, then we must also include $-1, -2, -3$, and -6.