UBC Grade 11/12 Solutions 1996

1. Check to see where the graph of $y = 2 - x$ is above the graph of $y = |3x + 1|$:

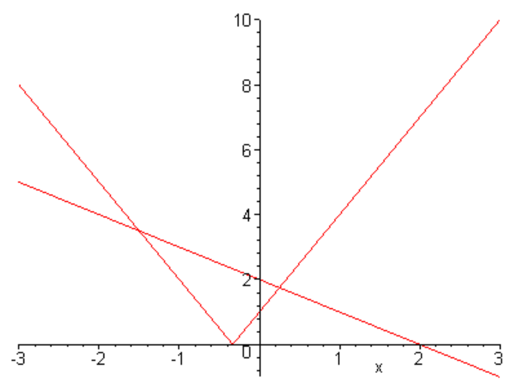

Figure 1: Graph for Question 1

We need the values of x between and including the intersection points. The intersection points satisfy $2 - x = -(3x + 1)$, or $x = -3/2$, and $2 - x = 3x + 1$, or $x = 1/4$. Thus, all solutions are given by the inequality $-3/2 \leq x \leq 1/4$.

2. Let x be the total number of goals scored against the goalie before the GAA drops to 2.54, and let y be the total number of games played at that point. Then, we are given two equations, namely,

$$\frac{x}{y} = 2.74, \frac{x + 3}{y + 3} = 2.54$$

Doing a straightforward substitution for x into the second equation leads us to $2.74y + 3 = 2.54y + 7.62$. Therefore, $y = 23.1$. Perhaps not surprisingly, y doesn't come out to be an integer. At this point, we can assume that the GAAs we were given were rounded to the nearest hundredth, and we can round our answers for x and y to the nearest integer *after* all the calculations are done. Substituting the found value for y into the first equation, we find that $x = 63.294$. Rounding $(x, y) = (63, 23)$, we now check the GAAs to see that they satisfy the given equations to two decimal places. $63/23 = 2.739...$ and $66/26 = 2.538...$ so our equations are satisfied. By the time the goalie's GAA has dropped to 2.54, 26 games have been played.

For the next part, let z be the number of games played for the rest of the season. Then, since a GAA of 1.5 is maintained for the rest of the season, the number of goals scored during the rest of the season is $1.5z$. An equation for the GAA at the end of the season is thus $(66 + 1.5z)/(26 + z) = 2$, and solving, we see that $z = 28$.

3. The train passes her 6 km before the crossing, which is equivalent to 1 hour of transit time for the cyclist. Since she is 50 minutes late, 10 minutes after the train passes her, the train has reached the crossing. Thus, the train travelled 6 km in 1/6 of an hour, and therefore it travels at a speed of 36 km/h.

4. We could have $x = 0$ provided that $y = 0$, but this is ruled out because we would then have to divide by zero in another equation. $xy = x/y$, therefore $y^2 = 1$, and $y = \pm 1$.
If $y = 1$, then $x/y = x - y$ implies that $x = x - 1$, which is impossible, since no number equals its own predecessor.
If $y = -1$, then $-x = x + 1$, implying that $x = -1/2$.
Thus, the only point satisfying the equalities is $(-1/2, 1)$.

5. Square both sides to get a quadratic equation:

$$x^2 = 1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + ...}}}$$

Notice that x appears on the right side of the second equation. Therefore, $x^2 - x - 1 = 0$. Use the quadratic equation to solve for x. $x = \frac{1}{2}(1 \pm \sqrt{5})$. We must reject the negative root because x is positive. Therefore, $x = \frac{1+\sqrt{5}}{2}$.

6. Observe that x is also in the denominator of the second term on the right side of the equation. We can thus rewrite:

$$x = 1 + 1/x \Rightarrow x^2 - x - 1 = 0$$

which is the same quadratic equation as in question (5). The answer is again $x = \frac{1+\sqrt{5}}{2}$, after we reject the negative root.

7. Let $r_n = \dfrac{x_{n+1}}{x_n}$. Then $r_n = 1 + \dfrac{1}{r_{n-1}}$. Consequently, if for some $k, r_k > 0$, then $r_{k+1} > 1$ and hence the limit $X > 1$.
$\lim_{n\to\infty} r_n = X = 1 + \frac{1}{X} \Rightarrow X^2 - X - 1 = 0$, and hence $X = \frac{1+\sqrt{5}}{2}$.
For X not to be $\frac{1+\sqrt{5}}{2}$, it is necessary that for each $k, r_k < 0$.
Let $s_k = -r_k > 0$ and $s_n = \dfrac{1}{s_{n-1}} - 1$. Then,

$$s_2 = \tfrac{1}{s_1} - 1 > 0 \Rightarrow \tfrac{1}{s_1} > 1 \Rightarrow 0 < s_1 < 1;$$
$$s_3 = \tfrac{1}{s_2} - 1 > 0 \Rightarrow 0 < s_2 < 1$$

This leads to

$$\frac{1}{s_1} - 1 < 1 \Rightarrow \frac{1}{2} < s_1 < 1 \Rightarrow \frac{1}{2} < s_2 < 1 \Rightarrow \frac{1}{s_1} - 1 > \frac{1}{2} \Rightarrow$$
$$\tfrac{1}{2} < s_1 < \tfrac{2}{3} \Rightarrow \tfrac{1}{2} < s_2 < \tfrac{2}{3}, \text{ etc.}$$

In this manner, one can show that in this case it is necessary that $s_n = s_1 = -X, n = 1, 2, \ldots$ Clearly here, $X = \frac{1-\sqrt{5}}{2}$. In particular, $b/a = \frac{1-\sqrt{5}}{2} = X$.

Thus, two cases arise: If $\dfrac{b}{a} = \dfrac{1-\sqrt{5}}{2}$, then $X = \dfrac{1-\sqrt{5}}{2}$; if $\dfrac{b}{a} \neq \dfrac{1-\sqrt{5}}{2}$, then $X = \dfrac{1+\sqrt{5}}{2}$.

8. Since AB is a diameter, $\angle ACB = 90°$. Consequently, $\triangle ACD$ and $\triangle ABC$ are similar. Thus, $\dfrac{x}{1} = \dfrac{x+1}{x}$, or $x = 1 + 1/x$ as before. $x = \dfrac{1+\sqrt{5}}{2}$. (Again!)

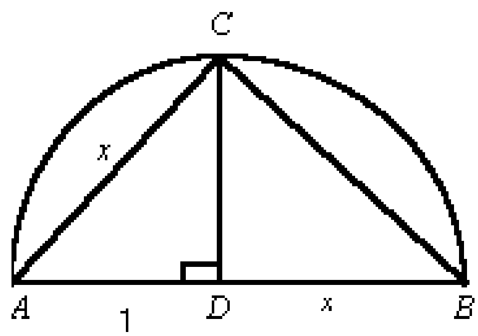

Figure 2: Dimensions for Question 8

Note: The number $\frac{1+\sqrt{5}}{2}$ is known as the Golden Ratio. This ratio appears in many places in mathematics (obviously!), including geometry (determining what shapes are aesthetically pleasing) and number theory (Fibonnaci numbers).

9. As the string is wound (figure on next page) from A to C, we are sweeping out an arc of radius 9 cm, over an angle of 120 degrees. As the string is wound from C to B, we are sweeping out an arc of radius 6 cm, over an angle of 120 degrees. As the string is wound from B to A, we are sweeping out an arc of radius 3 cm, over an angle of 120 degrees. Recall that the area of a sector of a circle is $\frac{\theta}{360}\pi r^2$. The total area swept out is thus

$$\frac{120}{360}\pi \times 9^2 + \frac{120}{360}\pi \times 6^2 + \frac{120}{360}\pi \times 3^2 = (27 + 12 + 3)\pi = 42\pi \text{ cm}^2$$

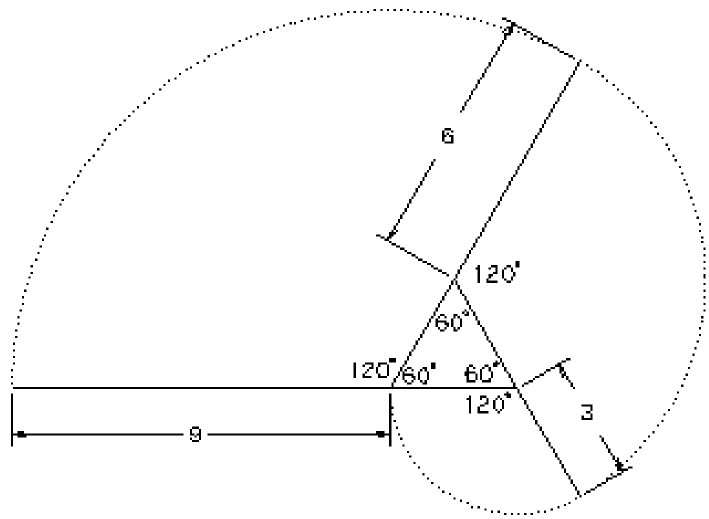

Figure 3: Winding a String Around a Triangle

10. By symmetry, the marked areas between the parabola and $y = 1$ are equal (see next page). Thus, the area in the square $[0, 1] \times [0, 1]$ to the right of the parabola is $1 - 2/3 = 1/3$. The function $x = y^2$ is just the first parabola turned 90 degrees clockwise. Thus, the area in the square $[0, 1] \times [0, 1]$ above $x = y^2$ is again $1/3$. Therefore, the area between the two parabolas is just the area of the square minus the areas outside of either of the parabolas. This is $1 - 1/3 - 1/3 = 1/3$ square units.

11. There are 2 choices for the first letter, 25 for the second, 24 for the third, and 23 for the fourth. Since we are allowed to have 3 or 4 letter length station names, there are $2(25)(24)(23) + 2(25)(24) = 28800$ possible station names.

12. We can score 150 points with three darts in three dart combinations: $100+25+25$ (there are 3 ways to do this with 3 darts), $75+50+25$ (6 ways), $50 + 50 + 50$ (1 way only). If all the darts thrown at the board land on it, we can calculate the probability of landing on a particular score segment using ratios of areas (we ignore units in the proceeding calculations because they will cancel). The area of a circle is equal to πr^2. (See diagram.)

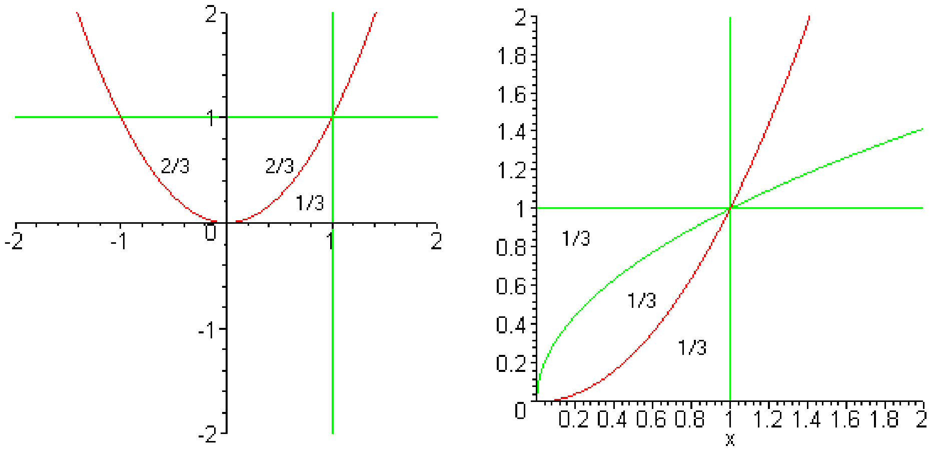

Figure 4: Parabolas and Areas

The total area is thus $7^2\pi = 49\pi$. The area of the 100 pt region is $1^2\pi$; the area of the 75 pt region is $(3^2 - 1)\pi = 8\pi$; of the 50 pt region, $(5^2 - 3^2)\pi = 16\pi$; of the 25 pt region, $(7^2 - 5^2)\pi = 24\pi$.
The corresponding probabilities of landing on each region are 1/49, 8/49, 16/49, 24/49 (check that these add to 1).
Going back to the three ways we can score 150 pts, we can finally calculate the probability of doing so:

$$3 \times 1 \times 24^2/49^3 + 6 \times 8 \times 16 \times 24/49^3 + 16^3/49^3 = 24256/117649 \approx 0.206.$$

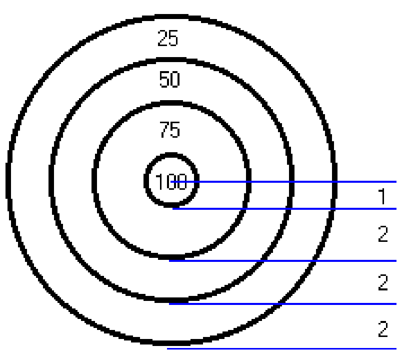

Figure 5: Dimensions of our Dart Board

13. A value of x for which F changes dramatically is $x = 0$. Thus, we want to check sections of the real line when $x + 1 = 0$ or $x - 1 = 0$, i.e. $x = -1, 1$. Make a table:

x	$x < -1$	$x = -1$	$-1 < x < 1$	$x = 1$	$x > 1$
$F(x+1)$	-1	0	1	1	1
$F(x-1)$	-1	-1	-1	0	1
$G(x)$	0	1	2	1	0

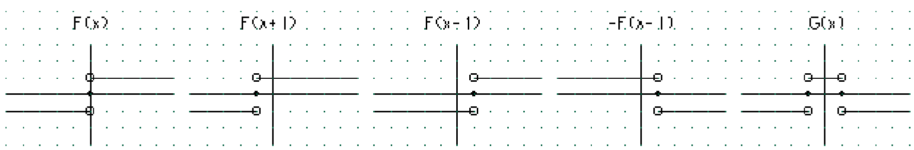

Figure 6: Graphs of Functions in Question 13

14. If $f(x) = f(x/2) + f(z)$, then $f(z) = f(x) - f(x/2)$. Since f is an increasing function, if $z \geq x/2$, then $f(z) \geq f(x/2)$. We can thus write $f(z)$ as $(1 + c)f(x/2)$, where c is positive or zero. Then, the original equation for f is $(1+c)f(x/2) = f(x)-f(x/2)$, or $f(x) = (2+c)f(x/2)$. This says that $f(x)$ is at least twice as much as $f(x/2)$.
This is clearly not true from the graph, so we can conclude that $z < x/2$.

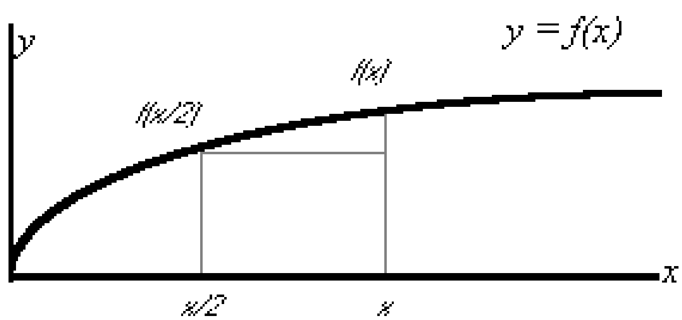

Figure 7: Graph for Question 14

15. a) From $A = B + C$, we can infer $\tan^{-1} x = \tan^{-1} y + \tan^{-1} z$ and therefore, $z = \tan(\tan^{-1} x - \tan^{-1} y)$. Since $y = x/2$, $z = \tan(\tan^{-1} x - \tan^{-1} \frac{x}{2})$. At this point, we make use of the trigonometric identity

$$\tan(a - b) = \frac{\tan a - \tan b}{1 + (\tan a)(\tan b)}$$

The expression for z found above simplifies to $\dfrac{x}{2 + x^2}$.

b) From the expression for z as a function of x, we now require $2z = x/2$. This implies that $\dfrac{2x}{2 + x^2} = \dfrac{x}{2}$, or $4x = 2x + x^3$, or $x^3 - 2x = 0$, or $x(x^2 - 2) = 0$. Thus, $x = 0, \pm\sqrt{2}$. But the question states that $A > 0$, so only $x = \sqrt{2}$ is a valid solution.

16. From the given inequality, we have $f(x) = 4x - \dfrac{2x^3}{3} \pm \dfrac{2x^3}{3}$, i.e. the $|error|$ in approximating $f(x)$ by $4x - \dfrac{2x^3}{3}$ is at most $\dfrac{2x^3}{3}$. Then,

$$f(1) = f(\tfrac{1}{2}) + f(\tfrac{1}{3}); f(\tfrac{1}{2}) = f(\tfrac{1}{4}) + f(\tfrac{2}{9}), f(\tfrac{1}{3}) = f(\tfrac{1}{6}) + f(\tfrac{3}{19})$$

Hence, $f(1) = f(\frac{1}{4}) + f(\frac{2}{9}) + f(\frac{1}{6}) + f(\frac{3}{19})$. We want $|error_{tot}| < 0.005$ in approximating $f(1)$; the error in approximating $f(\frac{1}{4})$ by $4x - \frac{2x^3}{3}$ is already $(\frac{2}{3})(\frac{1}{4})^3 = \frac{1}{96} > 0.005$. Hence we need to continue to "reduce":

$$\begin{aligned} f(\tfrac{1}{4}) &= f(\tfrac{1}{8}) + f(\tfrac{4}{33}) \\ f(\tfrac{2}{9}) &= f(\tfrac{1}{9}) + f(\tfrac{9}{83}) \\ f(\tfrac{1}{6}) &= f(\tfrac{1}{12}) + f(\tfrac{6}{73}) \\ f(\tfrac{3}{19}) &= f(\tfrac{3}{38}) + f(\tfrac{57}{731}) \end{aligned}$$

Thus,

$$\begin{aligned} &f(1) \\ &= f(\tfrac{1}{8}) + f(\tfrac{4}{33}) + f(\tfrac{1}{9}) + f(\tfrac{9}{83}) + f(\tfrac{1}{12}) + f(\tfrac{6}{73}) + f(\tfrac{3}{38}) + f(\tfrac{57}{731}) \\ &= 4(\tfrac{1}{8} + \tfrac{4}{33} + \tfrac{1}{9} + \tfrac{9}{83} + \tfrac{1}{12} + \tfrac{6}{73} + \tfrac{3}{38} + \tfrac{57}{731}) \\ &-(\tfrac{2}{3})[(\tfrac{1}{8})^3 + (\tfrac{4}{33})^3 + (\tfrac{1}{9})^3 + (\tfrac{9}{83})^3 + (\tfrac{1}{12})^3 + (\tfrac{6}{73})^3 + (\tfrac{3}{38})^3 + (\tfrac{57}{731})^3] \\ &+error \end{aligned}$$

where

$$|error| < (\frac{2}{3})[(\frac{1}{8})^3 + (\frac{4}{33})^3 + (\frac{1}{9})^3 + (\frac{9}{83})^3 + (\frac{1}{12})^3 + (\frac{6}{73})^3 + (\frac{3}{38})^3 + (\frac{57}{731})^3] < 0.00566$$

Hence more reduction is needed. $f(\frac{1}{8}) = f(\frac{1}{16}) + f(\frac{8}{129})$. Then,

$$\begin{aligned} &f(1) = f(\tfrac{1}{16}) + f(\tfrac{8}{129}) + f(\tfrac{4}{33}) + f(\tfrac{1}{9}) + \\ &f(\tfrac{9}{83}) + f(\tfrac{1}{12}) + f(\tfrac{6}{73}) + f(\tfrac{3}{38}) + f(\tfrac{57}{731}) \\ &= 4(\tfrac{1}{16} + \tfrac{8}{129} + \tfrac{4}{33} + \tfrac{1}{9} + \tfrac{9}{83} + \tfrac{1}{12} + \tfrac{6}{73} + \tfrac{3}{38} + \tfrac{57}{731}) \\ &-(\tfrac{2}{3})[(\tfrac{1}{16})^3 + (\tfrac{8}{129})^3 + (\tfrac{4}{33})^3 + (\tfrac{1}{9})^3 + (\tfrac{9}{83})^3 + (\tfrac{1}{12})^3 + \\ &(\tfrac{6}{73})^3 + (\tfrac{3}{38})^3 + (\tfrac{57}{731})^3] + error \end{aligned}$$

where $|error| < (\frac{2}{3})[(\frac{1}{16})^3 + (\frac{8}{129})^3 + (\frac{4}{33})^3 + (\frac{1}{9})^3 + (\frac{9}{83})^3 + (\frac{1}{12})^3 + (\frac{6}{73})^3 + (\frac{3}{38})^3 + (\frac{57}{731})^3] < 0.004674$ Consequently, we have shown that

$$f(1) = 3.14620776 \pm 0.004674$$

where we have guaranteed that our estimate is accurate to two decimals. $[\pi = 3.141592654...]$