UBC Grade 5-8 Solutions 1996

1. The total of all the marks before the error was spotted was: number of students $\times$ average mark $= 18 \times 86$.
After the error was found, the total is lowered by the error amount, $86-68 = 18$. Thus, the new, correct total is $18\times 86-18 = 18\times(86-1) = 18 \times 85$, and therefore the correct average is $18 \times 85/18 = 85$.
A quicker way of finding the answer would be to take note of the fact that the sum of the marks went down by 18, so that the average must have gone down by $18/18 = 1$.

2. The conversion factor is 7.5 mm rain per 8 cm snow, or 7.5 mm rain per 80 mm snow. Thus, the answer we seek is $500 \times (7.5 / 80) = 46.875$ mm of rain.

3. The maximum number of games played in a series is 7 (at this point, one team must have won 4 games). The loser of the series is eliminated and plays no more games. Thus, in the first round, there are 8 series; in the second, 4; in the third, 2; and, finally, in the fourth, 1. Since each series has a maximum of 7 games, the maximum number of playoff games is:

$$8 \times 7 + 4 \times 7 + 2 \times 7 + 1 \times 7 = 15 \times 7 = 105$$

An alternate way of viewing the problem is recognizing that 15 teams need to be eliminated, since there can only be one winner out of 16. Since each elimination requires the playing of a series, we need to 15 series, at a maximum of 7 games each, for $7\times 15 = 105$ games maximum.

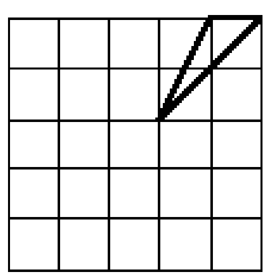
4. Let c = # of cars, b = # of bicycles, p = # of people. Then we are given 3 equations, namely:

$$\begin{array}{lllllll} (1) & 500c & + & 400b & & & = d \\ (2) & 600c & & & & & = d \\ (3) & & & 400b & + & 3000p & = d \end{array}$$

where d represents the capacity of the ferry. Then, from the first and third equations, we see that a car is worth six passengers, i.e. $c = 6p$. We can use this result with equation (2) to find that $600 \times 6p = d$, or in other words, the ferry can carry 3600 people.

5. Let $x =$ the number of goals let in after the first 30 games. We know that the number of goals let in during the first 30 games was 90, and that the total number of games played is at least 35. We are given that the final GAA was 2. The GAA has gone down, and we are asked to find out the minimum possible number of games played. The fastest way that the GAA can decrease is if no goals are let in at all for the remainder of the season, but we are told that there were only 5 shutouts. $90/35 \approx 2.57$, which is too big. The next fastest way of decreasing the GAA is to let in one goal per game. Let the additional number of goals scored after the first 90 be x. Then, we need $\frac{90+x}{35+x} = 2.00$ which we can simplify as $x = 20$. Thus, the minimum total games played during the season is $35 + 20 = 55$.
Note that we treated the GAAs as exact in the solution to this problem. What if they had actually been rounded? It turns out that even if the GAAs had been rounded, there would have been no confusion as to how many goals were scored or how many games were played, because the errors involved are small enough.

6. Notice that the figure is a triangle, and recall that the area of a triangle is $bh/2$. Let the base be the topmost side; then the height is 2 and the base is 1. The area of the triangle is thus 1 square unit. The total area is $5 \times 5 = 25$ square units; this implies that the region just outside the triangle has 24/25 or 96% of the area.



7. At every least common multiple of their departure intervals, the buses embark simultaneously from Granville and Georgia. Since the least common multiple of 12 and 20 is 60, the buses depart at the same time every hour on the hour. There are 17 "on the hours" between 7 a.m. and 11:30 p.m.

8. The ball is released from a height of 16 m, so it travels first a distance of 16 m. Then it proceeds to traverse each of the distances of 8, 4, and 2 m twice each. Finally, it travels a distance of 1 m only once. The total distance travelled is

$$16 + (2 \times 8) + (2 \times 4) + (2 \times 2) + 1 = 45\,\text{m}.$$

9. Cubes inside the first layer have no faces painted red. Cubes on the outmost layer have 1 face painted red only if they are not on an edge. Only the 4 cubes in the centre of each face satisfy this requirement. Thus, there are $6 \times 4 = 24$ cubes with only one face painted red.

10. The CDs were at least \$15, therefore the CDs must have cost at least $38 \times \$15 = \$570$. Let the missing digits be denoted by x and y, that is, let the total purchase price be $x29.2y$, where x and y are placeholders for *single*-digit numbers. The minimum total price calculated shows us that x must be one of 6, 7, 8, 9. We also know that the prices for each of the CDs was the same; thus, $x29.2y$ must be divisible by 38 (and thus must be divisible by 2, implying that y must be even). Trying each combination of valid x and y, we see that the only combinations that are divisible by 38 are \$629.28, and \$729.22. These results correspond to an individual CD price of \$16.56 or \$19.19.

11. All students can be put into four groups:
(i) Those who pass math and like it.
(ii) Those who don't pass math and like it.
(iii) Those who pass math and don't like it.
(iv) Those who don't pass math and don't like it.
We can devise a *Venn Diagram* to help us separate the students. Let's put all the students who like math in the L circle, and all the students who dislike it outside of the L circle. We also put the students who pass math in the P circle, while we put the students who don't pass

math outside of the P circle. We are told only one thing, that is, students who pass math like it. Thus, we have something in the region where P and L overlap; put an "x" here. Note that since students who pass math like it (or so we are told), the rest of the P circle is empty. Mark this with a "0." There are two regions of the diagram we are told nothing about. Mark these with a "?."

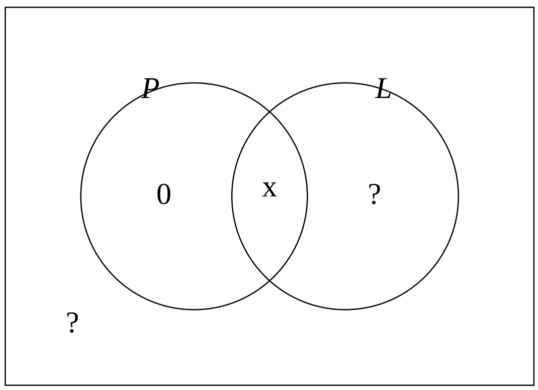


Figure 1: Venn Diagram

(a) We know nothing about the ? region within the L circle, so we cannot say with certainty that all students who like math are also within the P circle. We cannot determine the truth of (a) with certainty.
(b) The students who dislike math are outside of the L circle. For such a student to pass math, he/she must be within the P, but outside the L. Looking at our diagram, we see that this region is *empty*. Thus, (b) is true.
(c) It follows immediately from (b) that (c) is false.
(d) It follows immediately from (b) that (d) is true.
(e) This is a little tricky. It seems as if statement (e) should be true, since we have something in the region overlapped by P and L. However, we are told that students who pass math like it... We are *not* told that there exists even one such student. Thus, in actuality, the region marked "x" might actually be empty. We cannot determine the accuracy of statement (e) from the information we are given.

12. There is one "way" to get to the top of the topmost intersection. The next highest intersections can be reached in only one way each (take the paths left or right form the top). In fact, it can quickly be seen that the number of ways to reach any intersection on the diagram is the sum of the number of ways to get to the two intersections immediately above it:

$$\begin{array}{c} 1 \\ 1\ 1 \\ 1\ 2\ 1 \\ 1\ 3\ \ 3\ 1 \\ 1\ 4\ \ 6\ \ 4\ 1 \\ 1\ 5\ 10\ 10\ 5\ 1 \\ 1\ 6\ 15\ 20\ 15\ 6\ 1 \end{array}$$

This triangle is known as Pascal's Triangle. It is constructed using the summing rule mentioned above. The lowest point on the given diagram corresponds to the number 20 on Pascal's Triangle, so we conclude that there are 20 ways to get to the bottom of the square if we are only allowed to move downward.

13. The train passes Barbara 6 km before the crossing. This corresponds to 1 hour before Barbara reaches the crossing. 1 km later, corresponding to 10 minutes, Barbara would have reached the crossing had she not been late – at this time, the train has reached the crossing. Therefore, in 10 minutes = (1/6) hours, the train travelled 6 km. Therefore, the train moves at a speed of 36 kmph.

14. Let the distance from A to B be 1 unit. Then, the speed going downriver is 1/3, and the speed going upriver is 1/4. But, the speed downriver is also equal to the speed rowing (v_r) plus the water flow speed (v_f), and the speed upriver is equal to the speed rowing minus the water flow speed. Therefore, we have the equations:

$$1/3 = v_r + v_f$$
$$1/4 = v_r - v_f$$

So, $2v_f = 1/12$, or $1/v_f = 24$. This means that it takes 24 hours for the piece of wood to drift from A to B.

15. With just a single die, the probability that the "sum" is even or odd is obvious; they are just 1/2. Let's look at the probabilities for the sum of two dice. There are four possible outcomes that could interest us: EE, EO, OE, OO (where the Es and Os represent the even or oddness of the first and second die). For the cases EE and OO, the sum will be even; for the other two cases, the sum will be odd. Since the probability of getting even and the probability of getting odd on a single die is 1/2; each of these cases occurs with probability 1/4. Hence, the probability that the sum of two dice is even is P(EE) + P(OO) = 1/2 (implying that the probability for an odd sum is $1 - 1/2 = 1/2$). Now, imagine adding a third die to the sum of two. We have just found that the chance that the sum of two dice being even is 1/2, and we know that the chance that the third die is even is also 1/2. We again have four cases: (E)E, (E)O, (O)E, (O)O, where, this time, we are lumping the first two dice together within parentheses. The chance that three dice sum to an even number is therefore P((E)E) + P((O)O) = 1/2 again. We can continue this process for three dice and adding a fourth, grouping the first three together. Doing so, we can show that no matter how many dice we add, the probability that the sum is even is always 1/2. Thus, the chance that the sum of 5 dice is even is also 1/2.