UBC Grade 11/12 Solutions 1995

1. Rearranging, we get $0 \geq x^2 - 5x + 6 = (x-2)(x-3)$. Divide the real axis into sections with the roots, and let $f(x) = (x-2)(x-3)$.

x	$x < 2$	$x = 2$	$2 < x < 3$	$x = 3$	$3 < x$
$f(x)$	$+$	0	$-$	0	$+$

Thus, the desired interval is $2 < x < 3$.

2. $|3x| \geq |6-3x|$ simplifies to $|x| \geq |x-2|$. Thus, the distance from x to 0 is greater than or equal to the distance from x to the point 2. Hence the solution is $x \geq 1$.

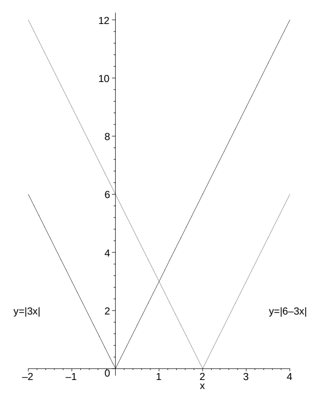

Figure 1: Graph for Question 2

3.

$$\begin{aligned}
\log_1(\log_2(\log_3(\log_4 x))) &= 0 \\
\log_2(\log_3(\log_4 x)) &= 1 \\
\log_3(\log_4 x) &= 2 \\
\log_4 x &= 3^2 \\
x &= 4^9 = 262144
\end{aligned}$$

4. $x^2 + x + \alpha = 0 \Rightarrow x = \frac{1}{2}(-1 \pm \sqrt{1 - 4\alpha})$. Check the discriminant $(1 - 4\alpha)$.
(i) $\alpha > 1/4$ – no real roots
(ii) $\alpha = 1/4$ – one real root
(iii) $\alpha < 1/4$ – two real roots
(iv) A quadratic can't have more than 2 real roots. Alternatively graph $y = x^2 + x = x(x+1) = f(x); f(-1/2) = -1/4$. Solution follows by translating graph up by 1/4.

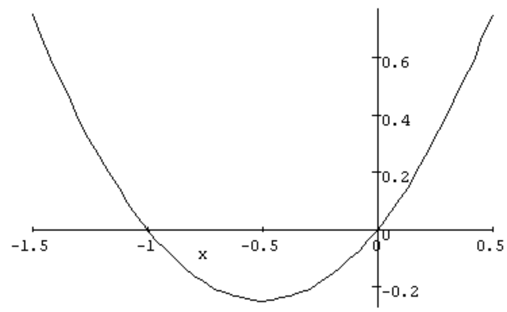

Figure 2: Graph for Question 4

5. Let D be the distance from Vancouver to Kelowna. Then the time of flight for a round trip is $T = \frac{D}{240} + \frac{D}{360}$. The average speed for the round trip is

$$\frac{2D}{T} = \frac{2}{\frac{1}{240} + \frac{1}{360}} = 288\,\text{kmph}$$

6. Since we are compounding 12 times over the year, we need to divide the interest rate by 12, and raise to the 12th power as shown: $(1 + 0.08125/12)^{12} = 1.084345052$. Hence the answer is 8.43%.

7. The circle is centred at the midpoint of the given diameter,

$$\left(\frac{5-1}{2}, \frac{-6+4}{2}\right) = (2, -1)$$

and has a radius that is half the length of the diameter,

$$\frac{1}{2}\sqrt{(5+1)^2+(-6-4)^2} = \sqrt{34}$$

Hence the equation of the circle is

$$(x-2)^2+(y+1)^2 = 34.$$

8. Sketch the graphs of $y = \sin x$ and $y = x/30$. (Remember that x is measured in radians.) The function $\sin x$ attains its maximum value of 1 when $x = \frac{\pi}{2} + 2k\pi, k = 0, \pm 1, \pm 2, ...$ Hence if $x > 0$ there are no more solutions when $\frac{1}{30}(\frac{\pi}{2} + 2k\pi) > 1 \Rightarrow k > \frac{15}{\pi} - \frac{1}{4} \Rightarrow k \geq 5$. There are two solutions for each $k = 0, 1, 2, 3, 4$. Since $\sin x$ and x are both odd functions of x it follows that there are 19 solutions. ($x = 0$ can only be counted once.)

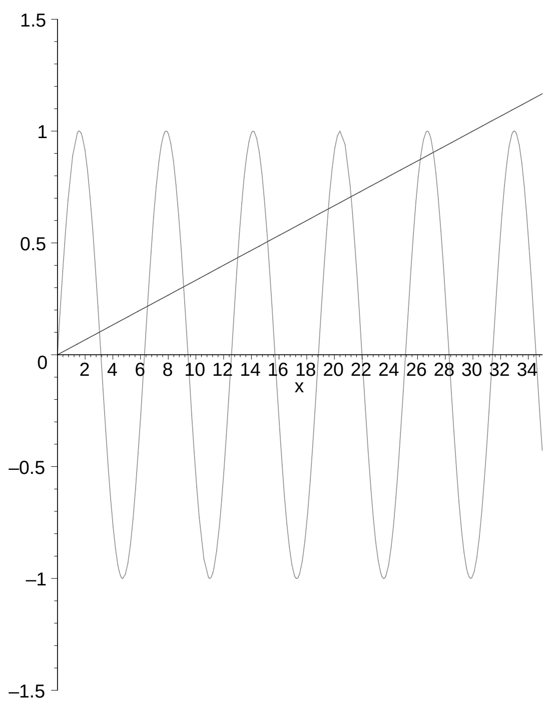

Figure 3: Graph for Question 8

9. The equation can be expressed as $x^3 + x = x(x^2 + 1) = 8(1 + 1/x^2)$. Clearly there are no solutions for $x \leq 0$. Graphically we see that there is one solution.

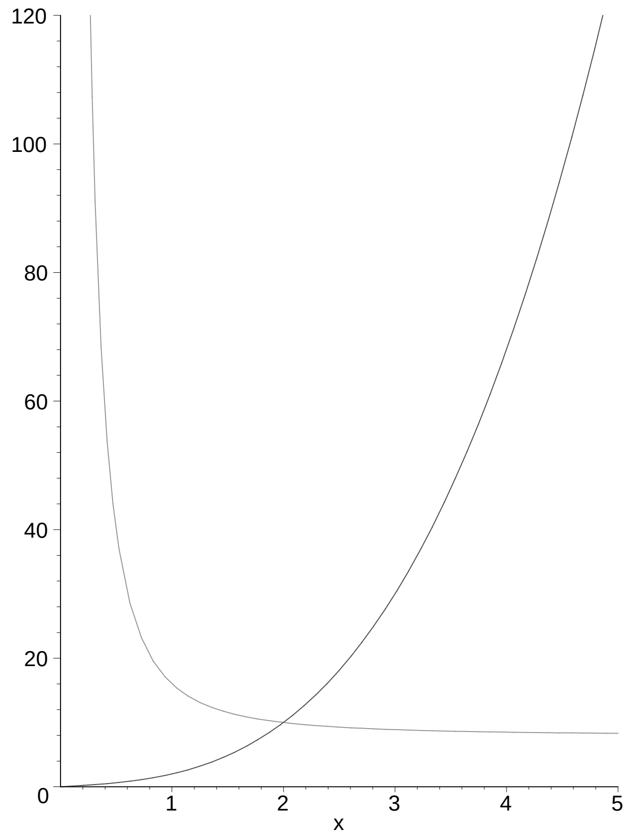

Figure 4: Graphs for Question 9

Alternatively, observe that the given equation can be factored as $(x^3 - 8)(x^2 + 1) = (x - 2)(x^2 + 2x + 4)(x^2 + 1) = 0$. Hence there is precisely one real solution. Which is the "better" solution and why?

10. $n^2 - n + 2 = n(n - 1) + 2$. Since $n(n - 1)$ is even, it follows that $n(n - 1) = 0$, or $n = 0, 1$.

11. We can do a few preliminary calculations to find a pattern. $[\sqrt[3]{1}] = [1] = 1, [\sqrt[3]{2}] = [1.259...] = 1, [\sqrt[3]{3}] = [1.4422...] = 1, ..., [\sqrt[3]{8}] = [2] = 2.$

It should be clear by now that $[\sqrt[3]{m}]$ is equal to the cubic root of the perfect cube that comes before m if m is not a perfect cube, or is equal to the cube root itself if m is a perfect cube. Imagine a job that pays you $\$[\sqrt[3]{m}]$ on the mth day. Then, we can interpret the question as asking on what day our average pay per day has reached \$2. That is, $[\sqrt[3]{1}] + [\sqrt[3]{2}] + \cdots + [\sqrt[3]{n}]$, our total pay up to day n, is equal to $2n$, where we view \$2 as our average pay per day.
On the first seven days, we are only making \$1 per day, since $[\sqrt[3]{m}] = 1$ for $1 \leq m < 7$ as was shown before. So far, we are seven dollars behind in total from making \$2 per day. From the eighth day to the 26th day, we are making \$2 per day, so we neither fall further behind nor gain ground on our targeted average salary. For the days from 27 to 63, we would make \$3 per day, an thus we will make up the \$7 that we lost on the first seven days, at a rate of \$1 per day. This requires seven days; so we need to work up to and including the 33rd day (there are seven days from 27 to 33). Since our average salary per day is always increasing, the solution is unique. Thus, $n = 33$.

12. Remember that there is one leap year every fourth year. 1992 was a leap year. Since $365/7 = 52 + 1/7$ (i.e. each non-leap year has 52 weeks plus a day), it follows that Jan 1 moves ahead two days in a year following a leap year and by one day in other years during the course of this century. Hence Jan 1 fell on a Saturday in 1994, on a Friday in 1993 and on a Wednesday in 1992 (leap year). Since $1993 - 1901 = 4 \times 23$, there were 23 leap years over that interval, and 23×3 "normal" years. Therefore, Jan 1, 1901 is $23 \times 2 + 69 = 115$ days behind Jan 1, 1993. But, $115 = 7 \times 16 + 3$; 115 days behind is equivalent to 3 days behind. Jan 1, 1993 was a Friday, we finally conclude that Jan 1, 1901 was a Tuesday.

13. If Richard Hickson looked at both cards and said he had at least one ace, then either he holds two aces (6 such combinations) or one ace and one non-ace $(4 \times 12 = 48)$ such combinations. Hence there are six chances out of 54 (i.e. $1/9$) that Richard Hickson holds at least two aces.

14. From the diagrams we see that $2l + 8r = 876/3$ and that $l = r\sqrt{3}$. Hence $r = \dfrac{146}{4+\sqrt{3}}$ mm.

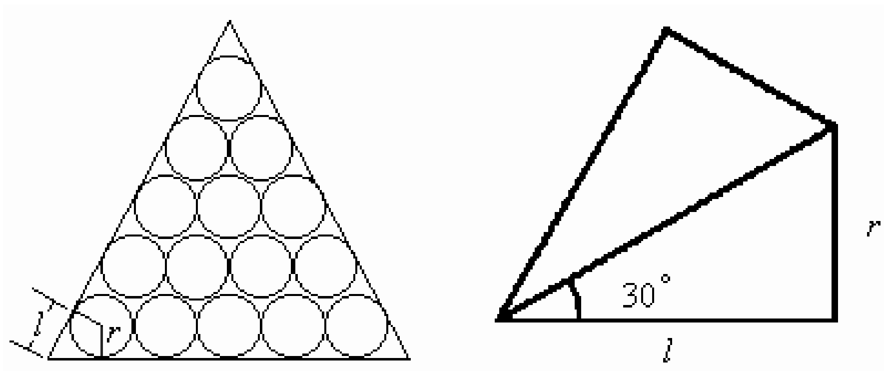

Figure 5: Geometry for Billiard Balls

15. (i) Let $a \le b \le c$ be the lengths of the sides of the triangle. Then, clearly, $l(p, P) < c < a + b$ since the length of any side of a triangle is less than the sum of the lengths of its other two sides. Consequently,

$$2l(p, P) < c + a + b \Rightarrow l(p, P) < S/2$$

since $S = a + b + c$.

(ii) A contradiction is shown by choosing an isosceles triangle with very acute base angles. Let the base length of the triangle be x. As the base angles approach zero, the perimeter S approaches $2x$, and $l(p, P)$ approaches x if we choose P to be one endpoint of the base, and p to be just inside the other endpoint. Then, the ratio $\dfrac{l(p, P)}{S} = \dfrac{x - \epsilon}{2x - \delta}$ for some ϵ, δ both greater than zero. As they approach zero, the ratio approaches 2, so that we can get arbitrarily close to 2. Therefore, we can find a (p, P) such that $l(p, P) < 0.49999S$ is false.