# UBC Grade 11/12 Solutions 1994

**1.** Rearranging, we get $0 \geq x^2 - 5x + 6 = (x-2)(x-3)$. Divide the real axis into sections with the roots, and let $f(x) = (x-2)(x-3)$.

| $x$ | $x < 2$ | $x = 2$ | $2 < x < 3$ | $x = 3$ | $3 < x$ |
|---|---|---|---|---|---|
| $f(x)$ | $+$ | 0 | $-$ | 0 | $+$ |

Thus $2 \leq x \leq 3$.

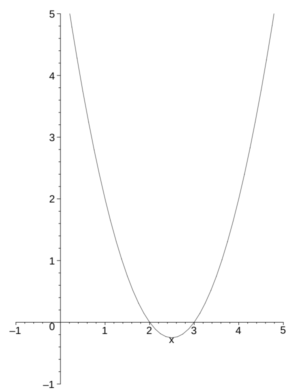


Figure 1: Graph for Question 1

**2.** $|3x| \geq |6 - 3x|$ implies $|x| \geq |x - 2|$. Thus, the distance from $x$ to 0 is greater than or equal to the distance from $x$ to the point 2. Hence the solution is $x \geq 1$. (Graph on next page.)

**3.** Observe the identity $\log_b x = \dfrac{\log_a x}{\log_a b}$, which we shall prove here: let $y = \log_b x$. Then, $b^y = x$, which means that $y \log_a b = \log_a x$, or $\log_b x = y = \dfrac{\log_a x}{\log_a b}$. We can use this identity to rewrite each term of the sum:

$$S = \frac{\log_2 2}{\log_2 2} + \frac{\log_2 3}{\log_2 2} + \frac{\log_2 4}{\log_2 2} + \cdots + \frac{\log_2 25}{\log_2 2} = \log_2(25!)$$

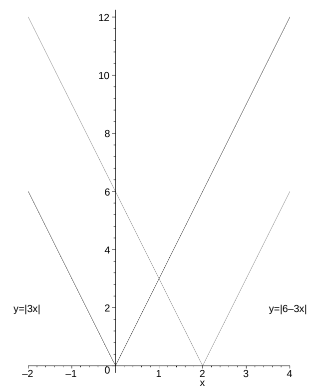


Figure 2: Graph for Question 2

**4.** Let $D$ be the distance from Vancouver to Kelowna. Then the time of flight for a round trip is $T = \left(\frac{D}{240} + \frac{D}{360}\right)$. The average speed for the round trip is

$$\frac{2D}{T} = \frac{2}{\frac{1}{240} + \frac{1}{360}} = 2(240)(360)/600$$
$$= 288\,\text{kmph}$$

**5.** An investment of $P$ dollars invested at an interest rate of $x\%$ compounded annually grows to a value of $(1+x)P$. An amount $P$ invested at 10% semi-annually collects interest twice per year, at half the specified rate. This means that after one year under this scheme, the investment grows to a value of $(1+\frac{0.10}{2})^2P$. Since we want the two investment schemes to be equivalent after one year, we need

$$\left(1 + \frac{0.1}{2}\right)^2 P = (1+x)P \Rightarrow 1.1025 = 1 + x \Rightarrow x = 10.25\%$$

**6.** The circle is centred at the midpoint of the given diameter,

$$\left(\frac{5-1}{2}, \frac{-6+4}{2}\right) = (2, -1)$$

and has a radius that is half the length of the diameter,

$$\frac{1}{2}\sqrt{(5+1)^2 + (-6-4)^2} = \sqrt{34}$$

Hence the equation of the circle is

$$(x-2)^2 + (y+1)^2 = 34.$$

**7.** Do a very rough sketch of the graphs $y = \sin x$ and $y = x/100$. (Remember that $x$ is measured in radians.) The function $\sin x$ attains its maximum value of 1 when $x = \frac{\pi}{2} + 2k\pi, k = 0, \pm 1, \pm 2, ....$ Hence if $x > 0$ there are no more solutions when the line is above $y = 1$. In terms of the periodic peaks of the sine function, this is $(\frac{\pi}{2} + 2k\pi)/100 > 1$ which simplifies to $k > \frac{50}{\pi} - \frac{1}{4}$, so no more intersections of the two graphs occur when $k > 15.66....$ So, the 15th peak of the sine function is where the last positive intersection is. There are two solutions for each $k = 0, 1, 2, ..., 15$. Since $\sin x$ and $x$ are both odd functions of $x$, it follows that there are 31 solutions. ($x = 0$ can only be counted once.)

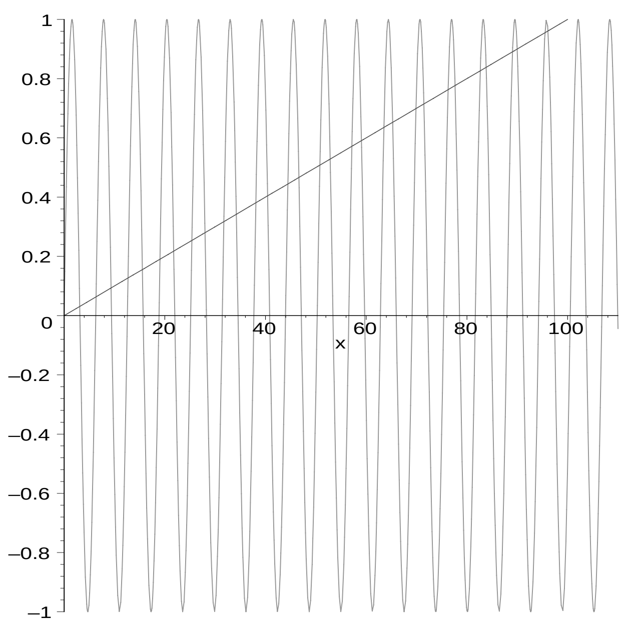


Figure 3: Graph for Question 7

**8.** The equation can be expressed as $x^3 + x = x(x^2+1) = 8(1+\frac{1}{x^2})$. Clearly, there are no solutions for $x \leq 0$. Graphically we see that there is one solution:

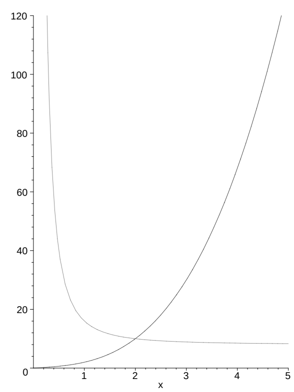


Figure 4: Graphs for Question 8

Alternatively, observe that the given equation can be factored as $(x^3 - 8)(x^2+1) = (x-1)(x^2+2x+4)(x^2+1) = 0$. Hence there is precisely one real solution. Which is the "better" solution and why?

**9.** We can do a few preliminary calculations to find a pattern. $[\sqrt[3]{1}] = [1] = 1, [\sqrt[3]{2}] = [1.259...] = 1, [\sqrt[3]{3}] = [1.4422...] = 1, ..., [\sqrt[3]{8}] = [2] = 2$. It should be clear by now that $[\sqrt[3]{m}]$ is equal to the cubic root of the perfect cube that comes before $m$ if $m$ is not a perfect cube, or is equal to the cube root itself if $m$ is a perfect cube. Imagine a job that pays you $\$[\sqrt[3]{m}]$ on the $m$th day. Then, we can interpret the question as asking on what day our average pay per day has reached \$2. That is, $[\sqrt[3]{1}] + [\sqrt[3]{2}] + \cdots + [\sqrt[3]{n}]$, our total pay up to day $n$, is equal to $2n$, where we view \$2 as our average pay per day.
On the first seven days, we are only making \$1 per day, since $[\sqrt[3]{m}] = 1$ for $1 \leq m < 7$ as was shown before. So far, we are seven dollars behind in total from making \$2 per day. From the eighth day to the 26th day, we are making \$2 per day, so we neither fall further behind nor gain ground on our targeted average salary. For the days from 27 to 63, we would make \$3 per day, an thus we will make up the \$7 that we lost on the first seven days, at a rate of \$1 per day. This requires seven days; so we need to work up to and including the 33rd day (there are

seven days from 27 to 33). Since our average salary per day is always increasing, the solution is unique. Thus, $n = 33$.

**10.** Remember that there is one leap year every fourth year. 1992 was a leap year. Since $365/7 = 52 + 1/7$ (i.e. each non-leap year has 52 weeks plus a day), it follows that Jan 1 moves ahead two days in a year following a leap year and by one day in other years during the course of this century. Hence, since we are given that it fell on a Saturday in 1994, Jan 1 fell on a Friday in 1993, on a Wednesday in 1992 (leap year) and on a Tuesday in 1991. Since $1993 - 1901 = 4 \times 23$, there were 23 leap years over that interval, and $23 \times 3$ "normal" years. Therefore, Jan 1, 1901 is $23 \times 2 + 69 = 115$ days behind Jan 1, 1993. But, $115 = 7 \times 16 + 3$; 115 days behind is equivalent to 3 days behind. Jan 1, 1993 was a Friday, we finally conclude that Jan 1, 1901 was a Tuesday.

**11.** If Richard Hickson looked at both cards and said he had at least one ace, then either he holds two aces (6 such combinations) or one ace and one non-ace ($4 \times 12 = 48$) such combinations. Hence there are six chances out of 54 (i.e. 1/9) that Richard Hickson holds at least two aces.

**12.** Let the length of $MN$ be $x$. If $a = b$, then it should be clear that we have a parallelogram, and thus $x = a$. Otherwise, we can extend the two non-parallel sides of the trapezoid to form a triangle as shown (next page). We have three similar triangles, the topmost of which is the smallest. The base of this triangle is given, it is equal to $a$. Let the height of this $\triangle OAB$ equal $ka$, a constant multiple of $a$. The area of the topmost triangle is thus $\frac{1}{2}ka^2$. The middle triangle, $\triangle OMN$, is similar to the top one. Let their scaling ratio be $c : 1$. Then, with a base of $x = ca$ and a height of $cka$, the area of this triangle is $\frac{1}{2}kc^2a^2 = \frac{1}{2}kx^2$. $\triangle ODC$ is also similar to the topmost one. Let the ratio between the larger one and the smaller one be $d : 1$. Then, the base of $\triangle ODC = b = da$ and the height is $dka$. The area is thus $\frac{1}{2}kd^2a^2 = \frac{1}{2}kb^2$. We are told that the area of the middle triangle minus the top one is equal to the area of the bottom one minus the middle one. This means that

$$\frac{k}{2}x^2 - \frac{k}{2}a^2 = \frac{k}{2}b^2 - \frac{k}{2}x^2$$

$$
\begin{aligned}
x^2 &= \frac{a^2 + b^2}{2} \\
x &= \pm\sqrt{\frac{a^2 + b^2}{2}}
\end{aligned}
$$

We must reject the negative root, because it is nonsensical. Notice that the expression is less than $b$, but greater than $a$; this makes physical sense, and reassures us that we have the correct answer.

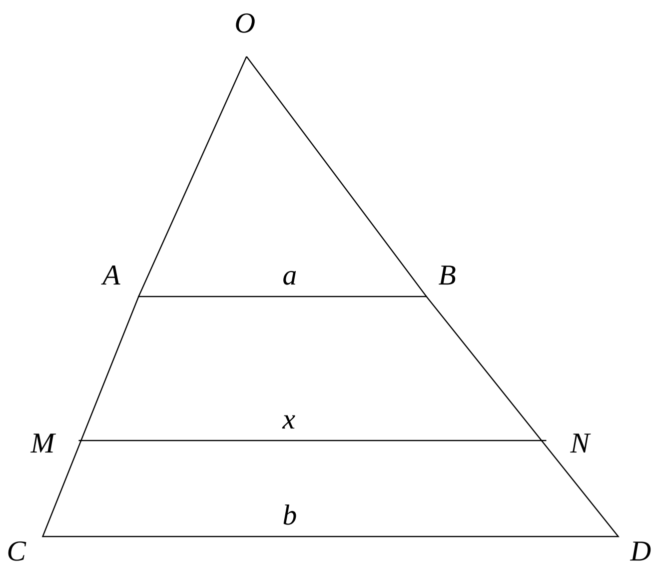


Figure 5: Triangles in Extended Trapezoid

**13.** From the diagrams we see that $2l + 8r = 876/3$ and that $l = r\sqrt{3}$. Hence $r = \dfrac{146}{4+\sqrt{3}}$ mm.

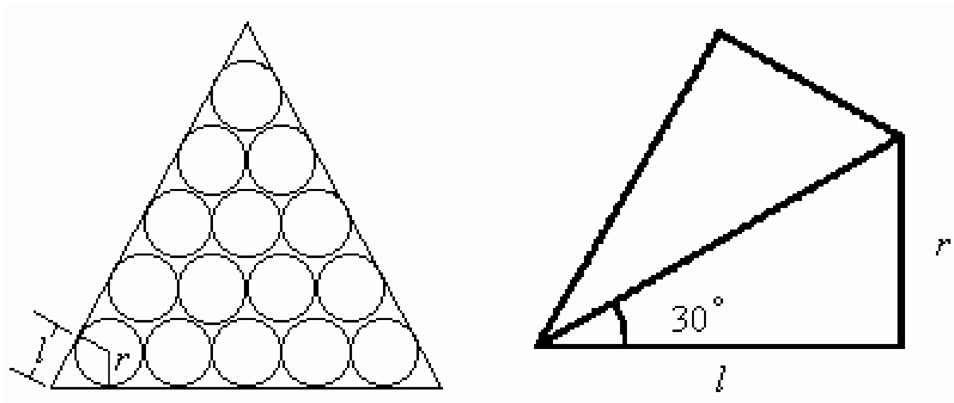


Figure 6: Geometry for Billiard Balls

**14.** From the information given, we know that the triangle lies within the rectangle $[0,4]\times[0,5]$. We can find the areas of the triangles in terms of $k$. $A_{rectangle} = 20$, $A_1 = \frac{1}{2}(2)k = k$, $A_2 = \frac{1}{2}(4)(3) = 6$, $A_3 = \frac{1}{2}(5)(4-k)$. We are given $A_4 = 8$. Then,

$$20 - k - 6 - \tfrac{1}{2}(5)(4-k) = 8 \Rightarrow k = \tfrac{8}{3}$$

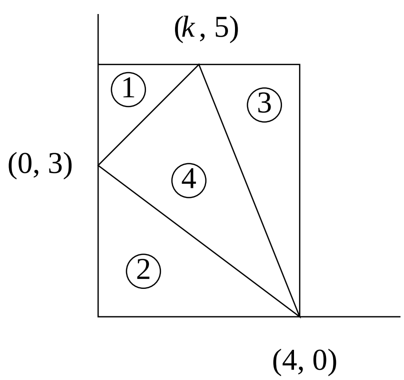


Figure 7: Coordinate Geometry for Question 14