UBC Grade 11/12 Solutions 1993

1. Let P be the profit in cents, c be the cost at which the tapes will be sold (in cents) and n be the projected number of tapes that will be sold. We are told that $n = 1000 + 50(100 - c) = 6000 - 50c$. The profit is $cn - 60n = n(c - 60)$, so we have a parabola,

$$\begin{aligned} P &= (c-60)(6000-50c) \\ &= -50(c^2 - 180c + 8100) - 360000 + 405000 \\ &= -50(c-90) + 45000 \end{aligned}$$

The parabola, which opens downward, has vertex (90,45000), so the most profitable cost setting would be 90 cents.

2. (a) $f(1) = 1, g(1) = 4$, so $f(1)/g(1) = \frac{1}{4}$
(b) $g(1) = 4, f(4) = 127$
(c) Isolate y, and then interchange x and y.

$y = 2x^3 - 1$
$y + 1 = 2x^3$
$x = \sqrt[3]{\dfrac{y+1}{2}}$
Interchanging x and y, we finally arrive at
$f^{-1}(x) = \sqrt[3]{\dfrac{y+1}{2}}$

3. First note that 1 to any power is 1. Thus, 1 is a solution. Now, suppose $x \neq 1$. $x^{(x^x)} = x^{x^2}$ then implies $x^x = x^2$ which in turn means that $x = 2$. Thus, $x = 1, 2$.

4. Assume that the hands move continuously, as opposed to discretely. The minute hand travels at a rate of 360 degrees over 60 minutes, or 6 degrees per minute. The hour hand moves at a rate of 360 degrees over

720 minutes (12 hours, not 24), or half a degree per minute. At 4:00, the minute hand points to 12 (0 degrees), while the hour hand points to 4 (120 degrees). Thus, letting t be the time in minutes after 4:00, an equation is

$$6t = 120 + 0.5t \Rightarrow t = \tfrac{240}{11}$$

At about 4:21.82, the minute hand overtakes the hour hand.

5. We are interested in finding the critical values of m for which the two graphs have only 1 intersection (the point of tangency).

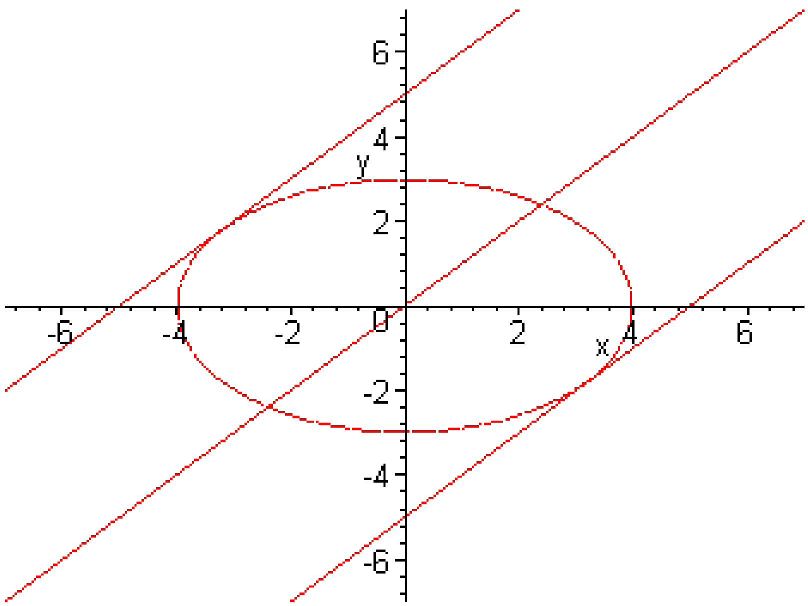

Figure 1: Ellipse and Lines

Substitute the equation of the line into the one for the ellipse. We have $\frac{x^2}{16} + \frac{x^2 + 2xm + m^2}{9} = 1$, which, simplified, is $25x^2 + 32mx + 16m^2 - 144 = 0$. Using the quadratic formula, we can solve for x. $x = \frac{-32m \pm \sqrt{(32m)^2 - 100(16m^2 - 144)}}{50}$. The discriminant determines the number of intersections. Expand the discriminant: $1024m^2 - 1600m^2 + 14400 = -576m^2 + 14400 = (120 + 24m)(120 - 24m) = 24^2(5 + m)(5 - m)$. The

critical value of $|m|$ is therefore 5.
If $|m| = 5$, we have a zero discriminant, and one intersection point.
If $|m| > 5$, we have a negative discriminant, and no intersection points.
If $|m| < 5$, we have a positive discriminant, and thus two intersection points.

6. For $x < 2$, we have $y = 2x + 2 - x = 2 + x$, and for $x \geq 2$, we have $y = 2x + x - 2 = 3x - 2$. A point of interest is $(x, y) = (2, 4)$; this is the "joint". Thus, for $y < 4$, $x = y - 2$, and for $y \geq 4$, we have $x = \dfrac{y+2}{3}$.

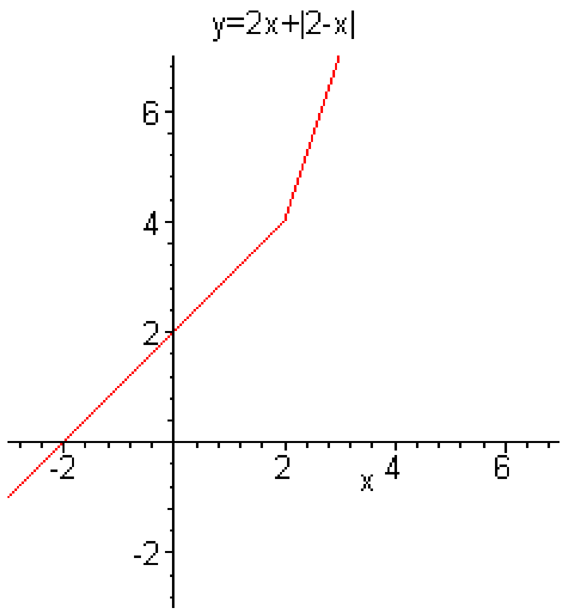

Figure 2: $y = 2x + |2 - x|$

7. We can rearrange the given equation in at least two useful ways.
$a^2 + b^2 + c^2 = bc + ca + ab \Rightarrow (a^2 - ab) + (b^2 - bc) + (c^2 - ca) = 0$
This in turn implies Equation $\alpha : a(a - b) + b(b - c) + c(c - a) = 0$
But, $a^2 + b^2 + c^2 = bc + ca + ab \Rightarrow (b^2 - ba) + (c^2 - cb) + (a^2 - ac) = 0$
This implies Equation $\beta : -b(a - b) - c(b - c) - a(c - a) = 0$
Add Equations α, β.
$a(a - b) - b(a - b) + b(b - c) - c(b - c) + c(c - a) - a(c - a)$
$= (a - b)^2 + (b - c)^2 + (c - a)^2 = 0$

Thus, we have a sum of squares adding to zero; the only way this can happen is if all the squares are zero. Therefore, $a = b = c$, and the triangle is equilateral.

8. Let's look at a generic parenthesized group of fractions. Each parenthesized group is of the form

$$\frac{1}{k+1} + \frac{2}{k+1} + \cdots + \frac{k}{k+1} = \frac{1+2+\cdots+k}{k+1} = \frac{k(k+1)}{2(k+1)} = \frac{k}{2}$$

which is a result known from arithmetic series. Now, notice that we are summing groups with $k = 1..99$, so that we may now write the original sum as

$$\sum_{k=1}^{99} \frac{k}{2} = \frac{1}{2}(1+2+\cdots+99) = \frac{99 \times 100}{2 \times 2} = 2475$$

9. We can solve this problem by listing all possible orderings of letters, and counting the ones that have no letters in the right place. We will number our letters 1, 2, 3, and 4, and an ordering of these letters we will display as a string of numbers, like 3241.

1234	1243	1324	1342	1423	1432
2134	**2143**	**2314**	2341	**2413**	2431
3124	**3142**	3214	3241	**3412**	**3421**
4123	4132	4213	4231	**4312**	**4321**

The orderings that have no numbers in the correct place are in bold. There are 9 such orderings.

The astute reader will realise that solving this type of problem by enumeration would be quite difficult if we added just one letter to the problem, as there are $5! = 120$ orderings of 5 letters. There is a general formula to solve this problem for any integer n, it is

$$n! \sum_{k=0}^{n} \left(\frac{(-1)^k}{k!} \right)$$

A derivation of this formula uses the principle of "inclusion-exclusion," which is explained in various mathematical textbooks and websites. (e.g.

`http://forum.swarthmore.edu/dr.math/problems/jarrad03.27.99.html`

`http://www.unc.edu/~rowlett/Math148/notes/incl_exclude.html`

– Links last visited 2000 August 03)

10. Let $Q(x)$ be the quotient upon dividing x^{100} by x^2+3x+2. Then, notice that the roots of x^2+3x+2 are -1 and -2. We can write $x^{100} = f(x) = (x^2+3x+2)Q(x)+P(x)$. Since the degree of $P(x)$ is no more than 1, let $P(x) = Ax+B$. We are now in a position to find the constant term of $P(x)$ by substituting the roots of the divisor into $f(x)$.

$$f(-1) = 1 = 0 \times Q(x) - A + B$$
$$f(-2) = 2^{100} = 0 \times Q(x) - 2A + B$$

Thus, $A = B - 1 = \frac{1}{2}(B - 2^{100})$. Solving, we find that B, the constant term of $P(x)$, is equal to $2 - 2^{100}$.

11. Draw a line parallel to AB through the point L.

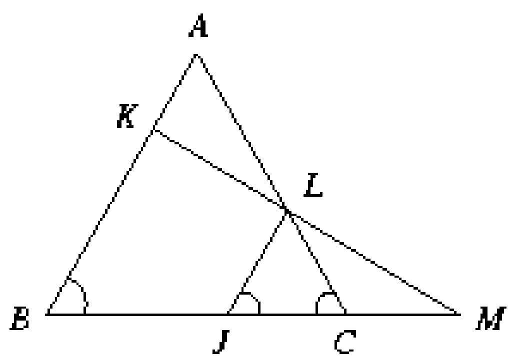

Figure 3: Geometry for Question 11

$\angle ABC = \angle LJM$ from the parallel lines theorem
$\angle JLM = \angle BKM$ from the parallel lines theorem
$\triangle BKM \sim \triangle JLM$ because of two equal angles (implying 3)
$\dfrac{KB}{JL} = 2$ because $KL = LM$, so the triangles are in a 2:1 ratio
$\angle ABC = \angle ACB = \angle LJC$. We are given that $\triangle ABC$ is isosceles
$\triangle JLC$ is isosceles. Therefore, $JL = LC$
From $\dfrac{KB}{JL} = 2$, we can now substitute to get $\dfrac{KB}{LC} = 2$.

12. Connecting the centres of the circles, we see that we have an equilateral triangle. Thus, the band spans $360 - 180 - 60 = 120°$ of each of the three circles; this is equivalent to the circumference of one of them,

$2\pi r = 2\pi$ m. It is also clear that there are three "straight" strips of length 2 radii = 2 m each, for another 6 m of length. The total length is $(6 + 2\pi)$ m.

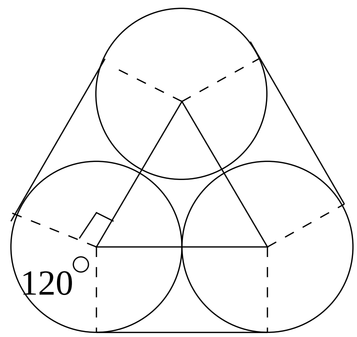

Figure 4: A Bundle of Pipes

13. Recall that the equation of a line in two dimensions can be found provided we have two distinct points. We can do this using the form

$$(y - y_1) + a(x - x_1) = 0$$

The case of a plane in three dimensions is analogous. Let our 3-D origin be located at the point T. Let the x-axis point along TU, the y-axis point along TW, and the z-axis point along TP. Let also the side length of the cube be 2. Then, we have three distinct points that lie on a unique plane, namely $(0, 0, 2), (1, 1, 0), (2, 1, 1)$. We can create three equations:

$$\begin{aligned} A(x - 0) + B(y - 0) + (z - 2) &= 0 \\ A(x - 1) + B(y - 0) + z &= 0 \\ A(x - 2) + B(y - 1) + (z - 1) &= 0 \end{aligned}$$

Keeping the Ax, By, z on the left side, we see that $2 = A + B = 2A + B + 1$, which can be solved easily. $A = -1, B = 3$. Thus, the equation of our plane is $-x + 3y + z - 2 = 0$. We are interested in the y-coordinate of the plane at the point $(2, \beta, 0)$. Putting this coordinate into the equation, we find that $\beta = \frac{2}{3}$, meaning $UX = \frac{2}{3}$, which implies $UV = \frac{4}{3}$, or, in other words, $\dfrac{UX}{XV} = \dfrac{1}{2}$.

14. Notice that we require that not both a and b are zero; this would cause a division by zero. If $a = 0$ and $b \neq 0$, then we can't have $x = \frac{k\pi}{2}$, where k is any integer, because these would make either the cosine or the sine function evaluate to zero, again causing an undefined quotient. For the other cases, cross multiply. We have
$ab\sin^2 x + b^2 \sin x + a^2 \sin x + ab = ab\cos^2 x + b^2 \cos x + a^2 \cos x + ab$
$ab(\sin^2 x - \cos^2 x) + (a^2 + b^2)(\sin x - \cos x) = 0$
$ab(\sin x - \cos x)(\sin x + \cos x) + (a^2 + b^2)(\sin x - \cos x) = 0$
$(\sin x - \cos x)[ab(\sin x + \cos x) + (a^2 + b^2)] = 0$
If either term is zero, then the product will be zero. The expression in parentheses is equal to zero when the sine and cosine functions are equal, this is the set of points $\frac{\pi}{4} + k\pi$, where k is an integer.
The bracketed term is harder to deal with. Rearranging the contents, we get to $\sin x + \cos x = -\dfrac{a^2 + b^2}{ab}$. For this equation to hold, we require that $\frac{a^2+b^2}{ab}$ be strictly less than 2, since a rough estimate of the maximum of $|\sin x + \cos x|$ is $1 + 1 = 2$. (With calculus, we can show that the maximum is actually $\sqrt{2}$.)

$$\begin{aligned} \frac{a^2 + b^2}{ab} &< 2 \\ a^2 + b^2 &< 2ab \\ a^2 - 2ab + b^2 &< 0 \\ (a - b)^2 &< 0 \end{aligned}$$

Clearly, this is impossible. So, we can have no other solutions to the given equation other than $\frac{\pi}{4} + k\pi$, where k is an integer.

15. $\cos(2x) = 1 - 2\sin^2(x) \Rightarrow \sin^2(x) = \dfrac{1}{2} - \dfrac{\cos(2x)}{2}$. So, we can replace the boundary $\sin^2(x)$. Notice that the area caught by $y = 0$ and $y = \cos(2x)$ "cancels" for $0 \leq x \leq \frac{\pi}{2}$ (see graph on next page). Thus, the area is just the area between $y = 0$ and $y = \frac{1}{2}$ from $x = 0$ to $x = \frac{\pi}{2}$. This is a rectangle; the area is $\frac{1}{2} \times \frac{\pi}{2} = \frac{\pi}{4}$.

16. We will make frequent use of two trigonometric identities:

$$\tan(2\theta) = \frac{2\tan(\theta)}{1 - \tan^2(\theta)}$$

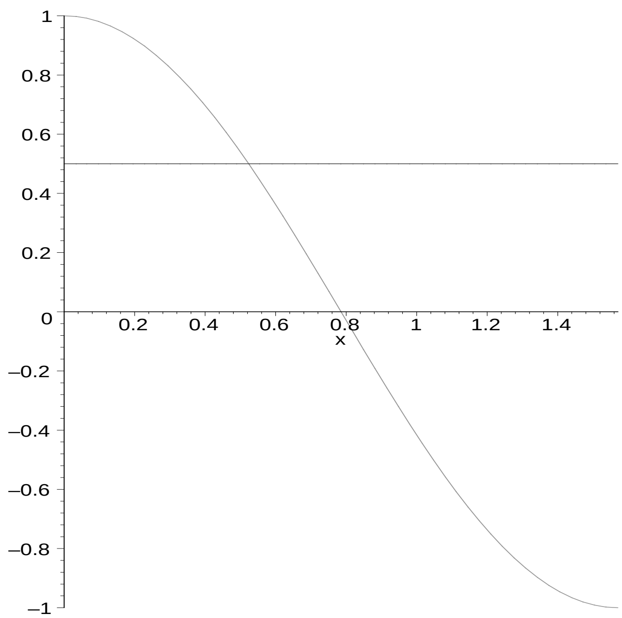

Figure 5: $y = \cos(2x)$ and $y = 1/2$

$$\tan(\theta + \phi) = \frac{\tan(\theta) + \tan(\phi)}{1 - \tan(\theta)\tan(\phi)}$$

We can derive the first identity as follows:

$$\tan(2\theta) = \frac{\sin(2\theta)}{\cos(2\theta)} = \frac{2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta} = \frac{2\sin\theta}{\cos\theta - \tan\theta\sin\theta}$$

$$= \frac{2\sin\theta}{\cos\theta(1 - \tan^2\theta)} = \frac{2\tan\theta}{1 - \tan^2\theta}$$

And the second one like this:

$$\tan(\theta + \phi) = \frac{\sin(\theta + \phi)}{\cos(\theta + \phi)} = \frac{\sin\theta\cos\phi + \sin\phi\cos\theta}{\cos\theta\cos\phi - \sin\theta\sin\phi}$$

$$= \frac{\dfrac{\sin\theta\cos\phi}{\cos\theta\cos\phi} + \dfrac{\sin\phi\cos\theta}{\cos\phi\cos\theta}}{1 - \dfrac{\sin\theta\sin\phi}{\cos\theta\cos\phi}} = \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}$$

(a) The situation is $\frac{\pi}{4} = \tan^{-1}(\frac{1}{m}) + \tan^{-1}(\frac{1}{n})$. We can take the tangent of both sides, with the help of the second identity. Thus, $1 = \frac{\frac{1}{m} + \frac{1}{n}}{1 - \frac{1}{m}\frac{1}{n}}$. Simplifying, we find that $mn - m - n = 1$, or $(m-1)(n-1) = 2$. Since the only factors of 2 are 1 and 2, the only (m, n) pairs that satisfy our equation are $(2, 3)$ and $(3, 2)$.

(b) We have $\frac{\pi}{4} = 2\tan^{-1}(\frac{1}{m}) + \tan^{-1}(\frac{1}{n})$. Again, take the tangent of both sides. Utilising the double-angle formula and the angle sum formula for the tangent function, we arrive at

$$1 = \frac{\dfrac{2m}{m^2-1} + \dfrac{1}{n}}{1 - \dfrac{2m}{m^2-1}\dfrac{1}{n}}$$

Grinding through some algebra leads to $n = f(m) = \dfrac{m^2+2m-1}{m^2-2m-1}$. The question concerns us with $m \geq 1$. Let's start listing the (m, n) pairs starting with $m = 1$:

m	1	2	3	4	5	6	7
n	-1	-7	7	$\frac{23}{7}$	$\frac{34}{14}$	$\frac{47}{23}$	$\frac{62}{34}$

It seems that the only (m, n) pair that satisfies our equation in this case is $(3, 7)$, because the function $f(m)$ seems to be a decreasing function of m, the limit as m goes to infinity of $f(m)$ is 1, and at $m = 7$, the value of $f(m)$ is already below 2. These facts are readily verifiable with the tools of calculus, but we are interested in finding a solution that makes use of tools in the Grade 12 curriculum.

Notice that we can rewrite $f(m)$:

$$\begin{aligned} \frac{m^2+2m-1}{m^2-2m-1} &= \frac{(m^2-2m-1)+(4m)}{m^2-2m-1} \\ &= 1 + \frac{4m}{m^2-2m-1} \\ &= 1 + \frac{4}{m-2-\frac{1}{m}} \end{aligned}$$

For $m \geq 1$ (the case we are told to consider), the denominator of the second term is an increasing function, so $\dfrac{4}{m-2-\frac{1}{m}}$ is a decreasing function. So, once the denominator is greater than 4, we are assured that $f(m) < 2$ and no higher values of m will lead to integer values of n. We want $m - 2 - \frac{1}{m} > 4$, which we can use the quadratic formula to solve. The largest root turns out to be $3 + \sqrt{10} \approx 6.16$, so for $m > 6$, we can conclude that larger values of m are inadmissible. Thus, the only (m, n) pair that satisfies our equation under the given constraints is $(3, 7)$.

(c) We have $\frac{\pi}{8} = \tan^{-1}(\frac{1}{m}) + \tan^{-1}(\frac{1}{n})$. Take the tangent of both sides. An exact value for $\tan\frac{\pi}{8}$ is not usually taught... We can find it by recalling the double-angle formula for the tangent function. Let $x = \tan\frac{\pi}{8}$.

$$\tan\tfrac{\pi}{4} = \tan(2 \times \tfrac{\pi}{8}) \Rightarrow 1 = \frac{2x}{1-x^2} \Rightarrow x^2 + 2x - 1 = 0 \Rightarrow x = -1 \pm \sqrt{2}$$

Since $\tan\frac{\pi}{8}$ is positive, we reject the negative root, and $\tan\frac{\pi}{8} = \sqrt{2} - 1$. Thus, we have $\sqrt{2} - 1 = \tan(\tan^{-1}\frac{1}{m} + \tan^{-1}\frac{1}{n})$. Going through algebra similar to part (a), we arrive at

$$\sqrt{2} - 1 = \frac{\frac{1}{m} + \frac{1}{n}}{1 - \frac{1}{m}\frac{1}{n}}$$

We have no (m, n) solutions in the positive integers, because the left side is irrational, while the right side is rational. So, we come to the conclusion that there are *no* integer pairs (m, n) such that our equation is satisfied.