UBC Grade 11/12 Solutions 1992

1. Let R be the revenue in cents, n be the number of transit users per day, and f be the fare, again in cents. Then, $R = nf$. We are told that an increase of 10 cents would decrease ridership by 10000 people; equivalently, an increase of 1 cent decreases ridership by 1000 people. Thus, $n = 250000 - 1000(f - 150) = 400000 - 1000f$. Substituting this into the equation for R, we see that

$$\begin{aligned} R &= (400000 - 1000f)f \\ &= -1000(-400f + f^2) \\ &= -1000(f - 200)^2 + 40000000 \end{aligned}$$

This is a downward-opening parabola, with vertex (200,40000000). Thus, the most advantageous fare increase for the transit system is \$0.50. This corresponds to a ridership of $250000 - 50 \times 1000 = 200000$, and a revenue of \$400000.

2. Points of interest for the given function are those where vertical asymptotes are located and where the roots are located. Factoring the numerator, we see that the roots reside at $x = 2$ and $x = 3$. Factoring the denominator, we see that the asymptotes reside at $x = 5$ and $x = 6$. Divide the real line into sections using these values of x. Let $f(x) = \dfrac{(x-2)(x-3)}{(x-5)(x-6)}$.

x	$x < 2$	$x = 2$	$2 < x < 3$	$x = 3$	
$f(x)$	$+$	0	$-$	0	
x	$3 < x < 5$	$x = 5$	$5 < x < 6$	$x = 6$	$x > 6$
$f(x)$	$+$	∞	$-$	∞	$+$

A less formal solution follows. $f(x) < 0$ either if (I)the numerator is

positive while the denominator is negative, or if (II)the numerator is negative while the denominator is positive. The numerator is negative when $2 < x < 3$, and positive when $x < 2$ or $x > 3$. The denominator is negative when $5 < x < 6$, and positive when $x < 5$ or $x > 6$. Case I occurs when $5 < x < 6$ and case II occurs when $2 < x < 3$.
Thus, we are interested in two regions, $2 < x < 3$, and $5 < x < 6$.

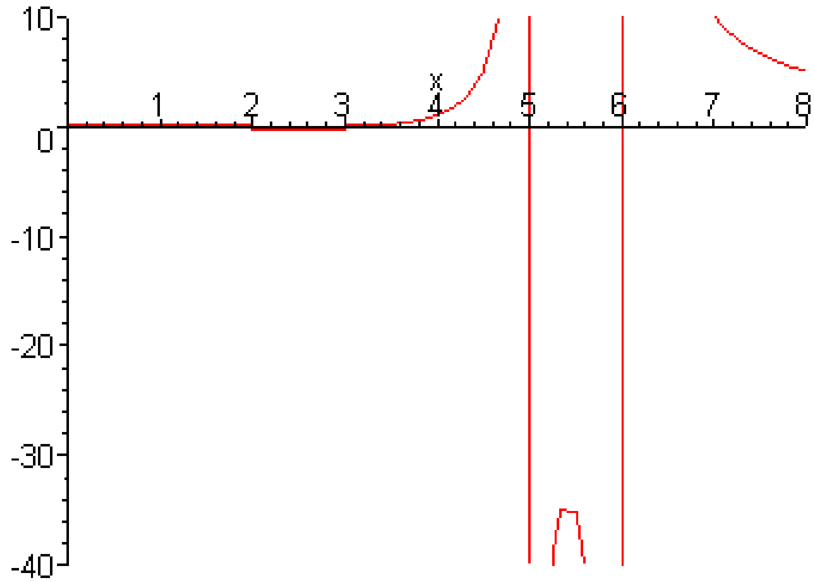

Figure 1: $\dfrac{x^2 - 5x + 6}{x^2 - 11x + 30}$

The region of interest $2 < x < 3$ is probably difficult to see, but on a computer plot or a graphics calculator with an appropriate zoom level, it will be obvious that it is a part of the desired solution.

3. $\left|\frac{3x-8}{2}\right| \geq 4 \Leftrightarrow |3x - 8| \geq 8 \Leftrightarrow |x - \frac{8}{3}| \geq \frac{8}{3}$. This means that the distance from x to $\frac{8}{3}$ is at least $\frac{8}{3}$, or,

$$x \leq 0 \text{ or } x \geq \tfrac{16}{3}$$

4. Let $y = \log_b N$. Then, $b^y = N$, and taking $\log_a$ of both sides yields $\log_a N = y \log_a b$, or

$$y = \log_b N = \frac{\log_a N}{\log_a b}$$

5. Pages 1 through 9 require only 1 digit each, for a subtotal of 9 digits. Pages 10 through 99 require 2 digits each, for a subtotal of $90 \times 2 = 180$ digits. Pages 100 through 999 require 3 digits each, and 900×3 is too many digits, so we know that the book has fewer than 999 pages. We are told that the printer has used 1980 digits; we have already used 189. This leaves 1791 digits for the 3-digit page numbers. $1791 \div 3 = 597$, so we have 597 3-digit page numbers. Starting from page 100, this means we end up at page 696.

6. Let the travelling distance be d in kilometers, the normal locomotive speed be v in km/h, and the normal travel time be t in hours. Then, we can construct 3 equations with 3 unknowns:
(a) $d = vt$
(b) $d = \frac{3}{5}v(t - 1 + 2) = \frac{3}{5}v(t + 1)$
(c) $d = 50 + v + \frac{3}{5}v\left(t - \frac{50}{v} - 1 + \frac{4}{3}\right)$
Equation (a) is straightforward and requires no explanation. Eq. (b) comes from the fact that the train travelled v km before the breakdown, causing a travel time of $t + 1$ hours at a reduced speed. Eq. (c) comes from the fact that the train travelled $v + 50$ km at normal speed, and the rest of the route at reduced speed; the total time elapsed at this reduced speed was the normal trip time minus the time used to travel the first $v + 50$ km, plus an 80 minute delay.
Noticing (a) and (b) are equal, and cancelling the common v's, we find that $t = 1 + \frac{3}{5}(t+1)$, or $t = 4$ h. Substituting this result into (a) and (c) yields $4v = v + 50 + \frac{3}{5}v(4 - \frac{50}{v} + \frac{1}{3})$ which has a solution $v = 50$ km/h. Finally, since $d = vt$, the distance between the start and the end of the route is $50 \times 4 = 200$ km.
Equation (c) is rather complicated; it is easy to make a mistake in formulating the correct equation. A less error-prone way of thinking of the problem would be to use the difference in breakdown delays. The time savings over 50 km without a breakdown is 2/3 of an hour. So, we have

$$\frac{50}{\frac{3v}{5}} - \frac{50}{v} = \frac{2}{3}$$

which has solution $v = 50\,\mathrm{km}$. Using equations (a) and (b), we can now find the route length without worrying about the troublesome equation (c).

7. 3 cases are distinguised. See graphs.
If $k > 0$, then we have an ellipse. In particular, if $k = 1$, we have a circle.
If $k = 0$, then we have $y = \pm 2$, two parallel lines.
If $k < 0$, then we have hyperbolas.

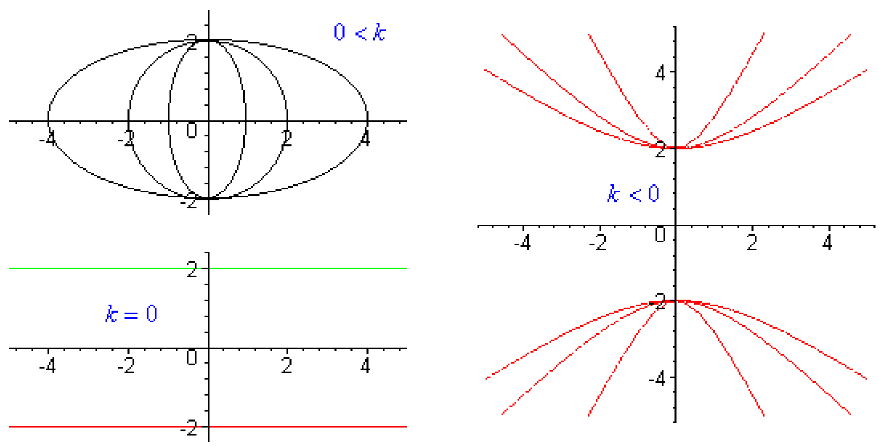

Figure 2: Graphs for Question 7

8. Use the cosine law. $4^2 = 5^2 + 6^2 - 2(5 \cdot 6)(\cos A)$, and $6^2 = 5^2 + 4^2 - 2(4 \cdot 5)(\cos C)$. Solving for A and C gives $A = \cos^{-1} \frac{3}{4}, C = \cos^{-1} \frac{1}{8}$. At this point, we can just use a calculator to divide the results for A and C to find that $C/A = 2$. However, we can use a cosine identity to reassure us that the ratio is *exactly* 2. Recall that $\cos(2\theta) = 2\cos^2(\theta) - 1$. Then, $\cos A = \frac{3}{4}$ implies $\cos(2A) = 2 \times \left(\frac{3}{4}\right)^2 - 1 = \frac{9}{8} - 1 = \frac{1}{8}$, which, as we have already found, is equal to $\cos C$. Recall that the cosine function takes on the value of $1/8$ in *two* places in the domain $\{x | 0 \leq x < 2\pi\}$. However, we can reject the angle in the fourth quadrant, because no triangle has an angle greater than 180 degrees. Thus, $2A = C$, or $C/A = 2$.

9. Try multiplying in front by $(1 - 4)$. This combines with $(1 + 4)$ to give $(1 - 4^2)$, which then combines with the next term to make another difference of squares. In the end, we have $(1-4^{32})(1+4^{32}) = (1-4^{64})$, but we must divide by $(1-4)$ because we multiplied by it at the beginning. Thus, the simplified result is $\dfrac{4^{64}-1}{3}$.

10. There is one "way" to get to the top of the topmost intersection. The next highest intersections can be reached in only one way each (take the paths left or right form the top). In fact, it can quickly be seen that the number of ways to reach any intersection on the diagram is the sum of the number of ways to get to the two intersections immediately above it:

$$\begin{array}{c}
1 \\
1\ \ 1 \\
1\ \ 2\ \ 1 \\
1\ \ 3\ \ 3\ \ 1 \\
1\ \ 4\ \ 6\ \ 4\ \ 1 \\
1\ \ 5\ \ 10\ \ 10\ \ 5\ \ 1 \\
1\ \ 6\ \ 15\ \ 20\ \ 15\ \ 6\ \ 1
\end{array}$$

This triangle is known as Pascal's Triangle. It is constructed using the summing rule mentioned above. The lowest point on the given diagram corresponds to the number 20 on Pascal's Triangle, so we conclude that there are 20 ways to get to the bottom of the square if we are only allowed to move downward.

11. Any multiple of 5 or its positive powers multiplied with an even number will end up adding another zero. Since there are many more even numbers than multiples of 5, we can ignore the counts of even numbers and just concern ourselves with multiples of 5 and it powers. First, notice that there are 20 multiples of 5 from 1 to 100, namely the integers 5, 10, 15, 20, 25, ..., 100. However, 5 "occurs" more than once in some of these numbers, i.e. in the multiples of 5^2. We need to count these numbers again. There are four: 25, 50, 75, 100. We need not worry about recounting higher powers of 5, because $5^3 = 125 > 100$, and higher powers don't occur in the list we are interested in. Hence, there are $20 + 4 = 24$ zeros at the end of 100! when it is written out in decimal form.
There is a function that can help us solve this problem algebraically.

Let $[x]$ denote the greatest integer function; that is, $[x]$ is the largest integer that is less than or equal to x. For example, $[3] = 3$ and $[5.7] = 5$. Then, we are interested in counting how many multiples of 5^n occur in the numbers from 1 to 100, where n is an integer greater than zero. The number of multiples of 5 from 1 to 100 is $\left[\frac{100}{5}\right]$; the number of multiples of 5^2 is $\left[\frac{100}{5^2}\right]$, the number of multiples of 5^3 is $\left[\frac{100}{5^3}\right]$, etc. Notice that by 5^3, no additional powers of 5 occur. Thus, the number of zeros in 100! is the number of multiples of 5 plus the "extra" power of 5 in the multiples of $5^2 = 25$,

$$\left[\frac{100}{5}\right] + \left[\frac{100}{5^2}\right] = 20 + 4 = 24$$

12. The probability of rolling a 6 is $\frac{1}{6}$, and of rolling a non-6 is $\frac{5}{6}$. We can examine each round that they play the game. In the first round, the chance that Carol throws the first 6 is $\left(\frac{5}{6}\right)^2 \frac{1}{6}$. For convenience, let this number be referred to as a. The chance that Carol throws the first 6 in the second round is $\left(\frac{5}{6}\right)^3 a$, since we require that the first round contained no 6s. The chance that Carol throws the first 6 in the third round is $\left(\frac{5}{6}\right)^6 a$. This is beginning to look suspiciously like a geometric series with base a and ratio $r = \left(\frac{5}{6}\right)^3$. Applying the geometric series formula, the infinite sum turns out to be $\dfrac{a}{1-r} = \dfrac{5^2}{6^3 - 5^3} = \dfrac{25}{91}$, or about 0.2747.

Another (perhaps easier) way of tackling this problem doesn't involve the use of geometric series. Let the probability that Alice wins be p. Then, the probability that Bob wins is $\left(\frac{5}{6}\right) p$, since we require that Alice does not roll a 6 in the winning round. Then, the probability that Carol wins is $\left(\frac{5}{6}\right)^2 p$ by similar reasoning. Someone must win the game eventually, so the sum of these three probabilities is 1. We have

$$\begin{aligned} p + \left(\tfrac{5}{6}\right) p + \left(\tfrac{5}{6}\right)^2 p &= 1 \\ \left(1 + \tfrac{5}{6} + \left(\tfrac{5}{6}\right)^2\right) p &= 1 \\ p &= \tfrac{36}{91} \end{aligned}$$

Thus, the probability that Carol wins is $5^2 \times p/6^2 = 25/91$.

13. We will show that $QV = 2\,\text{cm}$.
$QP \parallel RT$ and $QR \parallel PS$, because we are dealing with a parallelogram. Thus, $\angle PQT = \angle RTQ$ and $\angle QUP = \angle RQT$.
Therefore, $\triangle PQU \sim \triangle RTQ$. We also know that the large triangle is the small one scaled up by a factor of two, because we know the lengths of QU and QT to be in a 1:2 ratio. Hence, $QP = 0.5RT$.
$\angle QVP = \angle TVR$, and now we see that $\triangle PQU \sim \triangle RTU$. We know that $QP = 0.5RT$, and therefore these two triangles are also in a ratio of 1:2.
Thus, QV is half as long as VT, and since QT has a length of 6, QV must be $2\,\text{cm}$.

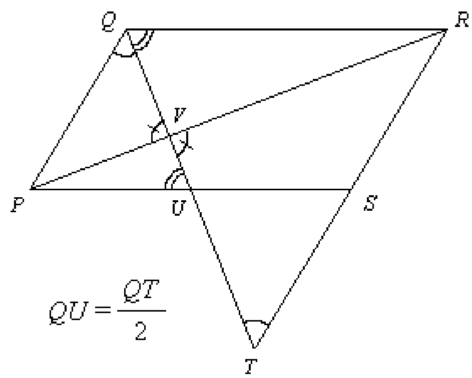

Figure 3: Similar Triangles

14. Drop a perpendicular from the intersection of the two arcs. Observe that we have a 30-60-90 triangle, and we can calculate the area of the sector as $\frac{60}{360} \times 2^2\pi = \frac{2\pi}{3}$. Half of the area under the archway is this area minus the area of the triangle: $\frac{2\pi}{3} - \frac{\sqrt{3}}{2}$. The area of two halves is the total area, this is

$$\left(\frac{4\pi}{3} - \sqrt{3}\right) \mathrm{m}^2$$

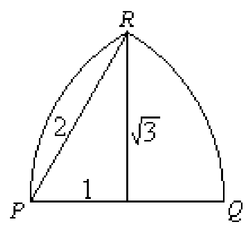

Figure 4: The Archway