UBC Grade 11/12 Solutions 1991

1. See diagram. The diagonals of a square meet at 90°. The radius of the circle is 1, so that the side length of the square is $\sqrt{2}$ by the Pythagorean Theorem. The area of the square is thus $(\sqrt{2})^2 = 2\,\text{cm}^2$.
Alternatively, observe that drawing in the diagonals of the square divides it into four right isosceles triangles with legs of length 1. We can rearrange these four triangles into two squares of side length 1 each, so we have a total area of $2 \times 1 = 2\,\text{cm}^2$.

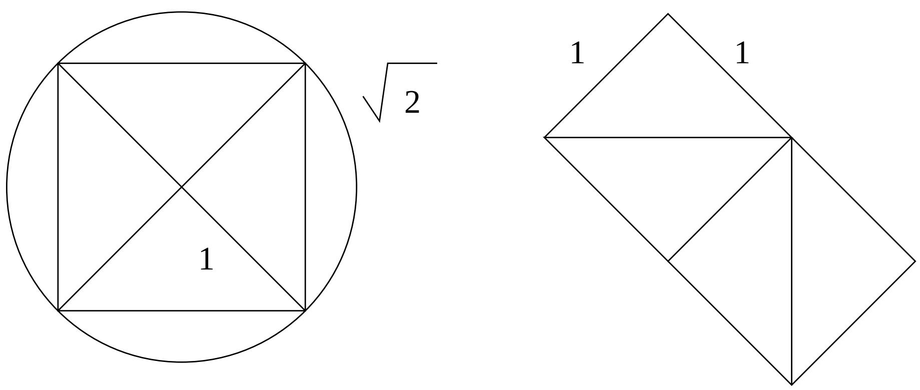

Figure 1: A Square Inscribed within a Circle

2. Let the first integer be denoted k. Then, to have n consecutive integers, we must have a list $k, k+1, k+2, ..., k+n-1$. The sum of these terms is $nk+\Sigma_{i=0}^{n-1} i = nk + \dfrac{n(n-1)}{2}$. The average is this sum divided by n. n clearly divides nk, so all that remains is to ensure that the second term ends up being an integer. This is guaranteed if n is an odd number, since we require $\frac{n-1}{2}$ to be an integer after the ns cancel.
We can also do this problem without knowledge of series. If n is odd, then the average is the number in the middle of the list (the median). If n is even, the average is halfway between the two numbers in the middle of the list, so it is not an integer in this case.

3. See diagram. Reflecting the smaller triangle across the table's edge, we see that we have a right triangle with legs 8 and 10. Thus,

$$\tan\alpha = \frac{8}{10} \Rightarrow \alpha = \tan^{-1}\frac{4}{5} \approx 38.66^\circ$$

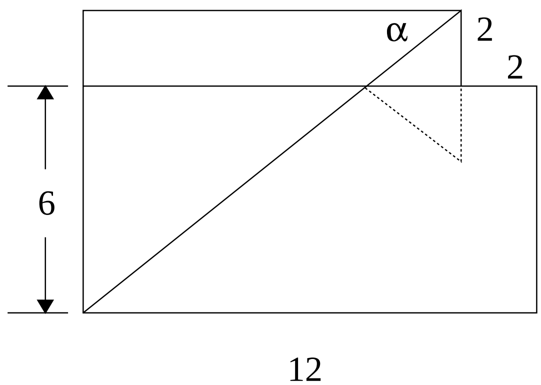

Figure 2: Our Pool Table

4. Let the radius of the base be r, and the height be h. Then, since we are cutting the cone at half its height, a cross section shows that the top part of the cone has half the proportions of the entire cone. The total volume is $V_{tot} = \dfrac{\pi r^2 h}{3}$, and the volume of the small upper cone is $V_{small} = \dfrac{\pi(\frac{r}{2})^2\frac{h}{2}}{3} = \dfrac{\pi r^2 h}{24} = \dfrac{V_{tot}}{8}$. Since we are given that the total volume was $24\,\mathrm{cm}^2$, the upper cone has a volume of $3\,\mathrm{cm}^2$, and the lower remainder therefore has a volume of $21\,\mathrm{cm}^2$.

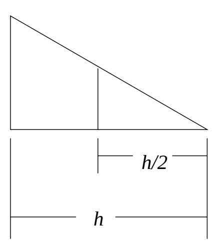

Figure 3: Cross Section of Cone

A faster way of solving this would be to notice that the whole cone has twice the linear dimensions of the top "half," so that it has $2^3 = 8$ times the volume. We are given that the whole cone has a volume of 24, so we can infer that the top part has a volume of $24/8 = 3$, leaving $21\,\text{cm}^2$ for the bottom "half."

5. One side length is x, the other is $\frac{A}{x}$. We have $B = x + 2\frac{A}{x}$, or $x^2 - Bx + 2A = 0 \Rightarrow x = \dfrac{B \pm \sqrt{B^2 - 8A}}{2}$. For a solution to exist, the discriminant must be at least zero, meaning that $B^2 \geq 8A$. As for the question of uniqueness, observe that it is possible for $\frac{B-\sqrt{B^2-8A}}{2}$ to be positive, so that we can have two solutions that make physical sense. However, if the discriminant is zero, then the solution is unique.

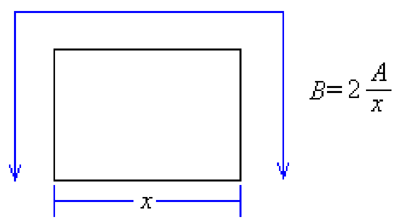

Figure 4: The Farmer's Field

6. Use the cosine law. It states that $c^2 = a^2 + b^2 - 2ab\cos C$. Since we are given $a^2 + b^2 > c^2$, $\cos C$ must be positive, i.e. $0° \leq C < 90°$ or $270° < C \leq 360°$. Since a triangle's interior angles must be in the domain $0° < \theta < 180°$, we can conclude that $0° < C < 90°$.
We can also solve this question geometrically. Imagine two legs (a and b) made out of rigid material and joined by a hinge, and a third elastic side, c (next page). Start with the two legs against one another. Slowly separate them; this stretches the elastic side c. As the angle C between a and b increases from 0 to $90°$, the length of c increases from 0 to $\sqrt{a^2 + b^2}$. Increasing the angle between a and b beyond $90°$ will increase the length of c. Since we are told that $a^2 + b^2 > c^2$, we want $0° < C < 90°$.

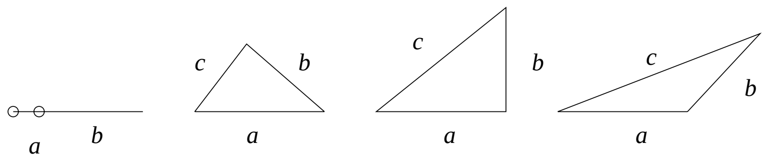

Figure 5: A Dynamic Triangle

7. Since "freshmen" is a sexist term, let us denote first-year students using "frosh." Since we are working with percentages, knowing the total frosh population is not important. Let's choose a convenient number, like 200, to be "equal" to the population. Then, we know that there were equal numbers of males and females among frosh, i.e. there were 100 of each. We are told that 0.83 of all frosh took a calculus course, and also that 0.43 of these were female. Thus, the percentage of female frosh taking a calculus course was $\frac{0.43 \times 0.83 \times 200}{100} = 71.38\%$. Similarly, the percentage of male frosh taking a calculus course was $\frac{0.57 \times 0.83 \times 200}{100} = 94.62\%$.

8. Let J and M denote the investments of John and Mary respectively at the end of one year, and let P_j, P_m denote their initial investments. Then, $J = (1 + 0.1) \times P_j = 1.1P_j$, and since Mary's interest is compounded twice per year, we must divide her interest rate by two and square it: $M = (1+0.05)^2 \times P_m = 1.1025P_m$. Hence, Mary earns 10.25 cents per dollar that she invests, while her brother earns just 10 cents per dollar invested. Thus, Mary ends up getting the better deal.

9. $mx - 1 + \frac{1}{x} \geq 0$ implies that $mx - 1 \geq \frac{-1}{x}$. Compare the graphs of $y = \frac{-1}{x}$ and $y = mx - 1$. We are interested in finding where the line is above the hyperbola.

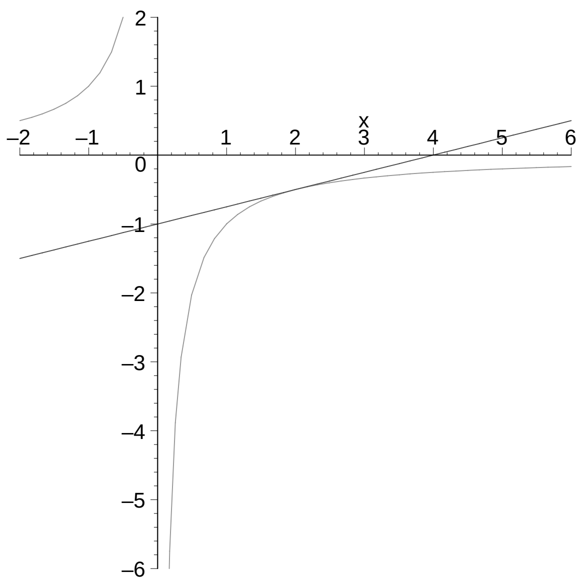

Figure 6: Find Critical m for Question 9

A critical value of m is when the hyperbola and the line have a single intersection; for slopes greater than or equal to this value, the line will be above the hyperbola for all positive x. $mx - 1 + 1/x \geq 0 \Rightarrow mx^2 - x + 1 \geq 0$. The only way that the two graphs have only one intersection is if this quadratic equation has only one root; i.e. it is a square of some linear function.

$$\begin{aligned} mx^2 - x + 1 &= 0 \\ m\left(x^2 - \frac{x}{m} + \frac{1}{m}\right) &= 0 \end{aligned}$$

From principles of completing the square, we know that for the expression above to be a square, we need $\left(-\frac{1}{2m}\right)^2 = \frac{1}{m}$. This implies that $m = 4m^2$, or $m = 1/4$. So, for $m \geq 1/4$, the inequality is satisfied.

10. Do a sketch of x^2 and $9\sin x$ on the same graph. The roots of the given equation are exactly the x-coordinates of the intersections of the two graphs.
It is quite obvious that one root is $x = 0$. There is also another root in the region $2.4 < x < 2.6$. Using a graphics calculator or computer

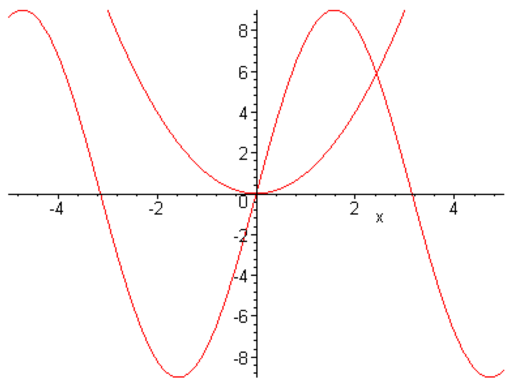

Figure 7: x^2 and $9 \sin x$

software, we can zoom in on the second intersection and read off the x-coordinate. We can also use the method of *bisection* to find this second root to two decimals. Let $f(x) = x^2$ and $g(x) = 9 \sin x$ (x is measured in radians).
The midpoint of [2.4, 2.6] is 2.5. $f(2.5) > g(2.5)$, so the root lies to the left of 2.5.
The midpoint of [2.4, 2.5] is 2.45. $f(2.45) > g(2.45)$, so the root lies to the left of 2.45.
The midpoint of [2.4, 2.45] is 2.425. $f(2.425) < g(2.425)$, so the root lies to the right of 2.425.
The midpoint of [2.425, 2.45] is 2.4375. $f(2.4375) > g(2.4375)$, so the root lies to the left of 2.4375.
The midpoint of [2.425, 2.4375] is 2.43125. $f(2.43125) > g(2.43125)$, so the root lies to the left of 2.43125.
The midpoint of [2.425, 2.43125] is 2.428125.
$f(2.428125) > g(2.428125)$, so the root lies to the left of 2.428125.
Notice now that we know that the root lies in the interval [2.425, 2.428125], which has a length 0.003125, which is means that *any* x in this region will be within 0.005 of the root, or, in other words, two decimal places. Thus, we can estimate the second root as the midpoint of this interval, $2.4265625 \approx 2.43$.

11. (a) This is just a straight line graph. Let the vertical variable be $w = \log_{10} y$, and the horizontal variable be $z = \log_{10} x$. Then, we just do a straight line plot of $w = 3 + 5z$.

(b) We can transform the equation:

$$
\begin{aligned}
&\log_{10} y = 3 + \log_{10} x^5 \\
&\log_{10} y = \log_{10}(10^3 \times x^5) \\
&y = 10^3 \times x^5 \\
&\log_5 y = \log_5 10^3 + 5 \log_5 x \\
&\log_5 y = 3 \log_5 10 + 5 \log_5 x
\end{aligned}
$$

Letting $w = \log_5 y$ and $z = \log_5 x$, this looks exactly like (a), except shifted up by a bit.

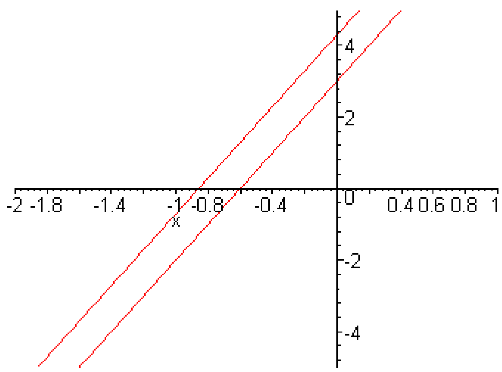

Figure 8: Plots for Question 11

12. The solution for r of the polynomial of degree two is quite simple, thanks to the quadratic formula. $r = \frac{1}{2}(1 \pm \sqrt{1 - 4\alpha})$, and we reject the negative root because it is the smaller one, and we are only interested in the larger. $r = \frac{1+\sqrt{1-4\alpha}}{2}$. The square-root function is well-known, and we can sketch the graph of r versus α as follows:

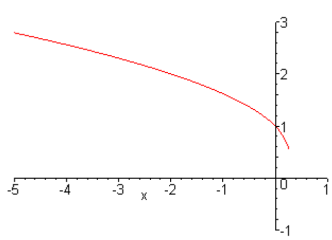

Figure 9: Plot for Quadratic Root

It is harder to find a function for the cubic root. $x^3 - x + \alpha = 0$ implies that $x^3 = x - \alpha$, so we want to find the x-coordinates of the places where the graphs of x^3 and $x - \alpha$ coincide. For this question, we are actually interested only in the largest root; thus, there will be a "jump" in our graph of r versus α at a critical value of α (see diagram). As the line moves up and becomes tangent to the part of the cubic above the x-axis, the largest root jumps from a negative value to a positive value. We wish to find this critical value of α.
This can be easily done using calculus (using derivatives to find local maxima or minima), but it can also be done using algebra.

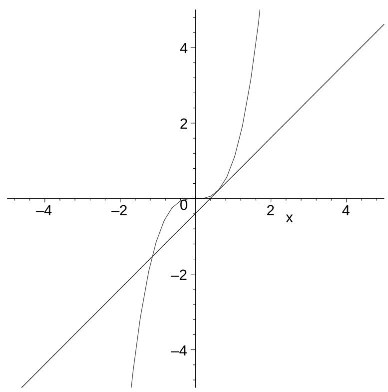

Figure 10: Intersections of Line and Cubic

For the line to become tangent to the cubic, we need a "double root" in our original equation. Notice that we have a cubic expression $x^3 - x + \alpha$ whose cubic term has a leading coefficient of 1. We can thus write it as $x^3 - x + \alpha = (x - s)(x - s)(x - t) = 0$, where s and t are the roots of the equation, and s is the double root that we require. Expanding the right side of our equation, we see that

$$x^3 - x + \alpha = x^3 - (2s + t)x^2 + (s^2 + 2st)x - ts^2$$

Since the left and right side are equal, the coefficients of the matching degree terms must be equal. From the x^2 terms, we see that $0 = 2s + t$, and from the x^1 terms, we see that $s^2 + 2st = -1$. The first equation says that $t = -2s$; substituting this into the second, we find

$$s^2 - 4s^2 = -1 \Rightarrow s = \pm\frac{1}{\sqrt{3}}$$

The negative root must be rejected; from our sketch, it is clear that the double root is positive. Thus, $t = -2s = \frac{-2}{\sqrt{3}}$ and, from matching the zero degree coefficients, we see that the critical value of α is $\alpha_c = -s^2 t = \frac{2}{3\sqrt{3}}$. So, for $\alpha \leq \alpha_c$, the largest root is positive, while for $\alpha > \alpha_c$, the largest root is negative.
$\alpha_c \approx 0.38$, and this value of α corresponds to the double root, so the largest root in this case is $s = \frac{1}{\sqrt{3}} \approx 0.58$. Note that if $\alpha = 0$, we can quickly factor the cubic to find that the largest root in this case is $r = 1$. Using these two points, we can make a rough sketch of the positive part of the graph. The negative part of the graph will be quite similar in appearance.

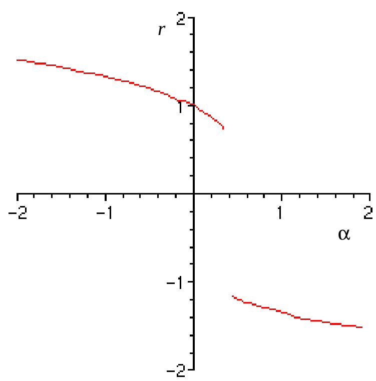

Figure 11: Plot for Cubic Root

13. $|x - a| = N|x - b|$. There are many cases of interest to this problem. If $a = b$, we have two subproblems. If $N \neq 1$ here, then the equation says that the distance from x to a is the same as N times the distance from x to a (remember that $a = b$ in this case). 0 is the only number that maps to itself when multiplied by a number other than 1. Thus, we must have $x = a$ when $a = b, N \neq 1$. If $N = 1$, then the equation says that the distance from x to a is the same as the distance from x to a, and therefore all x satisfy the equation.
The case of $a \neq b$ is not as straightforward, but can be done easily enough. Without loss of generality, assume that $a < b$. Referring to the given diagram will be helpful. For $a \neq b$, if $N = 1$, then we are told that the distance from x to a is equal to the distance from x to b; this is the definition of the midpoint between two points. Thus, we have

Figure 12: Number Line - Question 13

$x = \frac{a+b}{2}$. If $N > 1$, then we know that $|x-a| > |x-b|$; notice that this can happen in two places on our number line. A point s greater than b will allow $|x-a| > |x-b|$, while a point t between a, b but closer to b than a allows this too. A point s as described above gives $|s-a| = s-a$ and $|s-b| = s-b$. Thus, $|s-a| = N|s-b|$ implies that $s-a = N(s-b)$, which, solving for s, yields a solution $s = \frac{a-Nb}{1-N}$. A point t described as above gives $|t-a| = t-a$ and $|t-b| = b-t$. Therefore, $|t-a| = N|t-b|$ simplifies to $t = \frac{a+Nb}{1-N}$. Our last two subcases involve $0 < N < 1$. This says that $|x-a| < |x-b|$, which again can happen in two ways. We can either have a point u less than a, or a point v between a, b, but closer to a than b. Solving these two subcases is very similar to the work done just previously, $u = \frac{a-Nb}{1-N}, v = \frac{a+Nb}{1-N}$. We summarise the results:

a, b	N	Solutions x
$a = b$	$N \neq 1$	$x = a = b$
$a = b$	$N = 1$	$\{x \mid x \in \mathbf{R}\}$
$a \neq b$	$0 < N < 1$	$x = \frac{a-Nb}{1-N}, \frac{a+Nb}{1-N}$
$a \neq b$	$N = 1$	$x = \frac{a+b}{2}$
$a \neq b$	$N > 1$	$x = \frac{a-Nb}{1-N}, \frac{a+Nb}{1-N}$

where we assume $a < b$ if they are not equal.

14. Begin by looking at different orderings of the a small number of teams. For small numbers like 2, 3, and 4, we can solve this problem by listing all possible orderings of teams, and counting the ones that have no teams in the same place for both seasons. We can then divide by the number of arrangements of teams to find the probability that no teams finish in the same position. Subtracting this probability from 1, we will have found the probability that at least one team finishes in the same spot. We will number our teams 1, 2, 3, 4, ..., and an ordering of these teams we will display as a string of numbers, like 3241. Assume that

no ties are allowed.
For 2 teams, there are only two arrangements; one is correct (12), and the other has both teams in the "wrong" place (21). Thus, for the case of 2 teams, there is a 50% chance. There are $3! = 6$ ways of arranging 6 teams: (123, 132, 213, 231, 312, 321). Two of these arrangements have no teams in the correct rank. So, for 3 teams, four out of six arrangements have at least one team in the same place during both seasons, a probability of 67%.
The case for 4 teams begins to get tedious.

1234	1243	1324	1342	1423	1432
2134	**2143**	**2314**	2341	**2413**	2431
3124	**3142**	3214	3241	**3412**	**3421**
4123	4132	4213	4231	**4312**	**4321**

The orderings that have no numbers in the correct place are in bold. There are 9 such orderings. $1 - \frac{9}{24} = 62.5\%$
The astute reader will realise that solving by enumeration for even five teams would be quite difficult as there are $5! = 120$ orderings of 5 teams. We are asked to do 6 and 24 teams; if we're really patient, we can list all the cases as before and count up the orderings that don't have any team finishing in the same place. There is a general formula for n teams to count the number of arrangements that have no team in the "right" position, it is

$$n! \sum_{k=0}^{n} \left(\frac{(-1)^k}{k!} \right)$$

and, applying this equation, and then dividing by $n!$ (the number of possible arrangements), we can find that there is a 37% chance that at lease one team in the Smythe Division will finish in the same place for both seasons. It turns out that for 24 teams, the probability doesn't change much; to the nearest percent, it is still 37%. A derivation of this formula utilises a method called "inclusion-exclusion." This method is explained in various mathematical textbooks and websites. (e.g.
`http://forum.swarthmore.edu/dr.math/problems/jarrad03.27.99.html`
`http://www.unc.edu/~rowlett/Math148/notes/incl_exclude.html`
– Links last visited 2000 August 03)